Mathematical Statistics

Section 1

Preface:

I thank Allah Almighty who gave me a chance to serve the students in the field of Mathematics. I also thank the Holy Prophet Hazrat Muhammad (PBUH), who is forever a torch of guidance for humanity, without his goodwill nothing can be achieved. I am very grateful to my friends who encouraged me during the preparation of these notes. Further, I am thankful to my respected teacher **Prof. Dr. Hafiz Muhammad Khalid** for his kind support in this regard.

I feel myself lucky enough that my notes are popular all over Pakistan. Suggestions, comments, and indications of errors for the improvement will be highly appreciated.

I pray that my efforts will help the students in achieving their future objectives.

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- Punjab Group of College Talagang
- Barani Science College Talagang

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Section I

Q1 (a) : Probability theorems
(b) Numerical
Dr Khalid sb notes

Q2 : Total 4 parts
② parts discrete distributions theoretical

Q3 : ① Part discrete numerical.
① part Presentation Questions

Q4 (a) Property of normal Dis. theoritical
ⓑ Normal (numerical)

Section 2

Q5 : Total 3 Questions
1Q from regression correlation theoritical

Q6 : Baki 4 parts.
1 part from regression correlation numerical

Q7 : 1 part from Function of random variables
1 part from moment generating function of section I distributions
1 Part from section I distributions theoretical

Q8 : Both questions from three distributions

Q9. ① χ^2- distribution
② t-dis. and ③ F-distribution.
Theoretical

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Paper (IV-VI) option (i): Mathematical Statistics

NOTE: Attempt any FIVE questions selecting at least TWO questions from each section.

Section-I (4/9)

$e^{\mu t + \frac{1}{2}a^2 t^2}$

$e^{5t + 32t^2}$

$\mu = 5, \quad \frac{1}{2}a^2 = 32$

$a^2 = 64$

$a = 8$

$P(X = x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{a}\right)^2}$

$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-5}{8}\right)}$

Probability Distributions

- The postulates of probability
- Some elementary theorems
- Addition and multiplication rules
- Baye's rule and future Baye's theorem
- Random variables and probability functions

Discrete Probability Distributions

- Uniform, Bernoulli and binomial distribution
- Hypergeometric and geometric distribution
- Negative binomial and Poisson distribution

Continuous Probability Distributions

- Uniform and exponential distribution
- Gamma and beta distributions
- Normal distribution

Mathematical Expectations

- Moments and moment generating functions
- Moments of binomial, Hypergeometric, Poisson, Gamma, Beta and normal distributions

Section-II (5/9)

Functions of Random Variables

- Distribution function technique
- Transformation technique: One variable, Several variables
- Moment-generating function technique

Sampling Distributions

- The distribution of mean and variance
- The distribution of differences of means and variances
- The Chi-Square distribution
- The t distribution
- The F distribution

Regression and Correlation

- Linear regression

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Statistics:

Statistics are numerical statement of data in any department of inquiry.

Random Experiment.

An experiment which produces different results even though it is repeated a large number of times under essentially similar conditions, is called a random experiment

Sample Space.

It is the set of all possible occurence. It is denoted by S. For example, if we toss a single coin then head and tail are possible occurence.

Event. (E)

Any subset of a sample space is called an event.

Outcome.

The members or elements of a sample space are called outcomes of a sample space S.
For example if we roll a single die then the sample space S is given by

$S = \{1, 2, 3, 4, 5, 6\}$. Here 1, 2, 3, 4, 5, 6 are called outcome.

Probability.

Let S be a sample space and E be an event.
We define probability of E as

$$P(E) = \frac{\text{Number of outcomes in } E}{\text{Number of outcomes in } S} = \frac{n(E)}{n(S)} \; ; S \neq \phi.$$

Conditional Probability.

Let B be any given event. Then for any event A, the conditional probability of A is defined as

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \; , B \neq \phi.$$

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P(A/B) is read as, probability of A when B is g

Independent Events.

Two are more events are said to be independent i the probability of occurence of one event is not affected by the probability of occurence of other e

Mutually Exclusive Events.

Two events A and B of a single experiment are said to be mutually exclusive or disjoint if and only if they cannot both occur at the same time. If the two events can occur at the same time, they are not mutually exclusive.

Exhaustive Events.

Events are said to be collectively exhaustive, whe the union of mutually exclusive events is the entire sample space.

Equally Likely Events.

Two events are said to be equally likely, when one event is as likely to occur as the other.

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THEOREMS AND IMPORTET PROBLEMS:

1- Show that the probability of an impossible event is zero., i.e $P(\phi)=0$

Since an impossible event contains no sample points, therefore the certain event 'S' and the impossible event ϕ are mutually exclusive.

Also

$$S \cup \phi = S$$

$$\Rightarrow \quad P(S \cup \phi) = P(S)$$

$$P(S) + P(\phi) = P(S)$$

$$P(\phi) = 0 \quad \text{Required}$$

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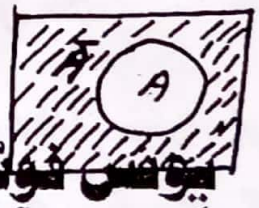
2- If 'A' is an event such that its complementary event $\bar{A}$ is disjoint to 'A', then $P(\bar{A}) = 1 - P(A)$

Hence deduce that $P(A) \leq 1$ and $P(\phi)=0$

Since A and $\bar{A}$ are mutually exclusive events and also



$$\bar{A} \cup A = S$$

$$P(\bar{A} \cup A) = P(S)$$

$$P(\bar{A}) + P(A) = 1$$

$$P(\bar{A}) = 1 - P(A) \quad \because S \text{ is a sure event} \quad \therefore P(S) = 1$$

Also

$$P(A) = 1 - P(\bar{A})$$

So $P(A) \leq 1 \quad \because P(\bar{A}) \geq 0$

Also we know $\phi = \bar{S}$

$$\Rightarrow P(\phi) = P(\bar{S}) = 1 - P(S)$$

$$= 1 - 1 = 0$$

Hence $P(\phi) = 0$ Required

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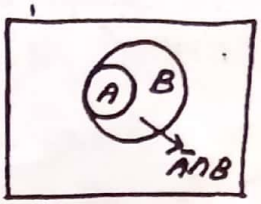
3- If 'A' and 'B' are two events such that $A \subset B$, then $P(A) \leq P(B)$

$\because$ $A \subset B$

$\therefore$ $A \cap B = A$



Since $A \cap B$ and $\bar{A} \cap B$ are mutually exclusive events and

$$B = (A \cap B) \cup (\bar{A} \cap B)$$

$$P(B) = P[(A \cap B) \cup (\bar{A} \cap B)]$$

$$P(B) = P(A \cap B) + P(\bar{A} \cap B)$$

$$P(B) = P(A) + P(\bar{A} \cap B)$$

$$\Rightarrow P(B) \geq P(A) \quad \because P(\bar{A} \cap B) \geq 0$$

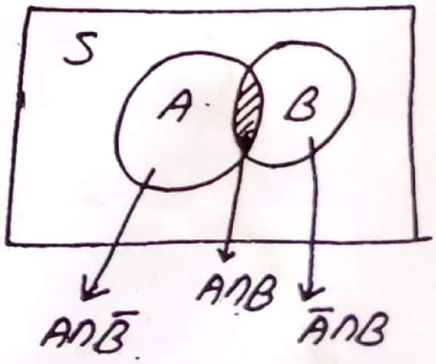
Hence '$P(A) \leq P(B)$'

4- If 'A' and 'B' are any two events, then

(i) $P(A \cap \bar{B}) = P(A) - P(A \cap B)$

(ii) $P(\bar{A} \cap B) = P(B) - P(A \cap B)$

From figure, we can see that $A \cap \bar{B}$ and $A \cap B$ are mutually-exclusive events and also



$$A = (A \cap \bar{B}) \cup (A \cap B)$$

$$P(A) = P[(A \cap \bar{B}) \cup (A \cap B)]$$

$$P(A) = P(A \cap \bar{B}) + P(A \cap B)$$

$$\Rightarrow P(A \cap \bar{B}) = P(A) - P(A \cap B)$$

Similarly, we can prove

$$P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

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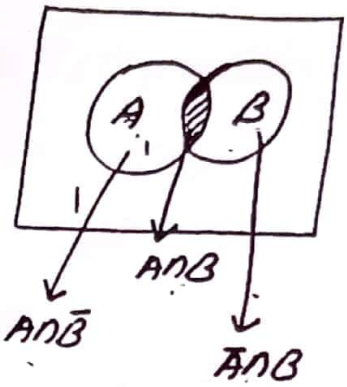
5- For any two events 'A' and 'B', prove that

$$P[(A\cap\bar{B})\cup(\bar{A}\cap B)] = P(A)+P(B)-2P(A\cap B)$$

From venn diagram, it is clear that $A\cap\bar{B}$ and $\bar{A}\cap B$ are mutually exclusive events, so



$$P[(A\cap\bar{B})\cup(\bar{A}\cap B)]$$

$$= P(A\cap\bar{B}) + P(\bar{A}\cap B)$$

$$= P(A) - P(A\cap B) + P(B) - P(A\cap B)$$

Hence using 4-

$$P[(A\cap\bar{B})\cup(\bar{A}\cap B)] = P(A)+P(B)-2P(A\cap B)$$

6- Addition law:

If 'A' and 'B' are any two events then $P(A\cup B) = P(A)+P(B)-P(A\cap B)$

From above figure, it is clear that $A\cap\bar{B}$ and $A\cap B$ are mutually exclusive and their union is equal to 'A', that is

$$(A\cap\bar{B})\cup(A\cap B) = A$$

$$P[(A\cap\bar{B})\cup(A\cap B)] = P(A)$$

$$P(A\cap\bar{B}) + P(A\cap B) = P(A)$$

$$P(A\cap\bar{B}) = P(A) - P(A\cap B) \quad \text{--- (1)}$$

Also from above venn diagram, we have $A\cap\bar{B}$ and B are mutually exclusive events. Moreover

$$A\cup B = (A\cap\bar{B})\cup B$$

$$P(A\cup B) = P(A\cap\bar{B}) + P(B)$$

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$$P(A\cup B) = P(A) - P(A\cap B) + P(B) \quad \text{using ①}$$

$$P(A\cup B) = P(A) + P(B) - P(A\cap B) \quad \text{Required.}$$

Addition law 7 - for three events 2013

For any three events A, B and C,

$$P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A\cap B) - P(A\cap C) - P(B\cap C) + P(A\cap B\cap C)$$

Consider

$$P(A\cup B\cup C) = P(A\cup D) \quad \text{where } B\cup C = D$$

$$= P(A) + P(D) - P(A\cap D)$$

$$= P(A) + P(B\cup C) - P[A\cap(B\cup C)]$$

$$= P(A) + P(B) + P(C) - P(B\cap C) - P[(A\cap B)\cup(A\cap C)]$$

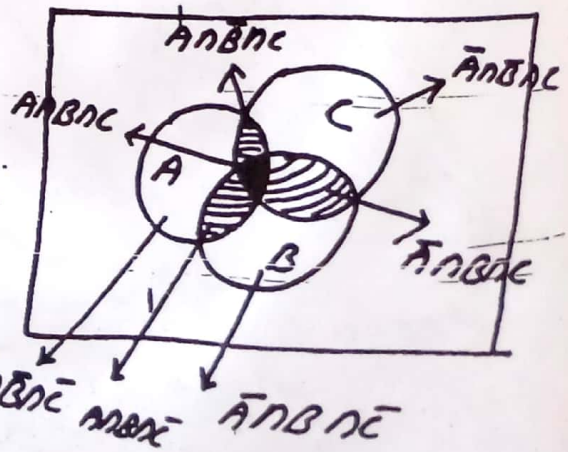
$$= P(A) + P(B) + P(C) - P(B\cap C) - P(A\cap B) - P(A\cap C) + P(A\cap B\cap C)$$

Required.

8- Prove that by using Venn diagram.

$$P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A\cap B) - P(B\cap C) - P(A\cap C) + P(A\cap B\cap C)$$

From Venn diagram, it is clear that the seven events $A\cap\bar{B}\cap\bar{C}$, $\bar{A}\cap\bar{B}\cap C$, $\bar{A}\cap B\cap\bar{C}$, $A\cap B\cap\bar{C}$, $A\cap\bar{B}\cap C$, $\bar{A}\cap B\cap C$, $A\cap B\cap C$ are mutually exclusive. Also their union is the union $A\cup B\cup C$.



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$$A \cup B \cup C = (A \cap \bar{B} \cap \bar{C}) \cup (\bar{A} \cap B \cap \bar{C}) \cup (\bar{A} \cap \bar{B} \cap C)$$
$$\cup (A \cap B \cap \bar{C}) \cup (A \cap \bar{B} \cap C) \cup (\bar{A} \cap B \cap C) \cup (A \cap B \cap C)$$

$$\Rightarrow P(A \cup B \cup C) = P(A \cap \bar{B} \cap \bar{C}) + P(\bar{A} \cap B \cap \bar{C}) + P(\bar{A} \cap \bar{B} \cap C)$$
$$+ P(A \cap B \cap \bar{C}) + P(A \cap \bar{B} \cap C) + P(\bar{A} \cap B \cap C) + P(A \cap B \cap C) \quad \text{—— (1)}$$

Also from figure.

$$\left.\begin{aligned} A \cap \bar{B} \cap \bar{C} &= A - A \cap B - A \cap C + A \cap B \cap C \\ \bar{A} \cap B \cap \bar{C} &= B - A \cap B - B \cap C + A \cap B \cap C \\ \bar{A} \cap \bar{B} \cap C &= C - A \cap C - B \cap C + A \cap B \cap C \end{aligned}\right\} \text{—— (2)}$$

$$\left.\begin{aligned} A \cap B \cap \bar{C} &= A \cap B - A \cap B \cap C \\ A \cap \bar{B} \cap C &= A \cap C - A \cap B \cap C \\ \bar{A} \cap B \cap C &= B \cap C - A \cap B \cap C \end{aligned}\right\} \text{—— (3)}$$

Putting (2) & (3) in (1), we get
(After applying probability)

$$P(A \cup B \cup C) = P(A) - \cancel{P(A \cap B)} - \cancel{P(A \cap C)} + \cancel{P(A \cap B \cap C)}$$
$$+ \cancel{P(A \cap B)} - \cancel{P(A \cap B \cap C)} + \cancel{P(A \cap B \cap C)} + P(B)$$
$$- P(A \cap B) - \cancel{P(B \cap C)} + \cancel{P(A \cap B \cap C)} + \cancel{P(A \cap C)}$$
$$- \cancel{P(A \cap B \cap C)} + P(C) - P(A \cap C) - P(B \cap C)$$
$$+ P(A \cap B \cap C) + \cancel{P(B \cap C)} - \cancel{P(A \cap B \cap C)}$$

Thus

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$
$$- P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Required.

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9- If $A_1, A_2, \text{——}, A_n$ are mutually exclusive events, then prove that

$$P(A_1 \cup A_2 \cup \text{--} \cup A_n) = P(A_1) + P(A_2) + \text{—} + P(A_n)$$

OR $$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

We will prove it by Mathematical induction. Let 'S' be a sample space and A_1, A_2 are any two events in 'S'. Let n, n_1, n_2 are the number of sample points in S, A_1 and A_2 respectivily. Then $A_1 \cup A_2$ contains $n_1 + n_2$ sample points if both are mutually exclusive.

Then $$P(A_1 \cup A_2) = \frac{n_1 + n_2}{n}$$

$$= \frac{n_1}{n} + \frac{n_2}{n}$$

$$= P(A_1) + P(A_2)$$

This shows that result is true for $n=2$. Next, we suppose that our result is true for $n=K$.

i.e $$P(A_1 \cup A_2 \cup \text{--} \cup A_K) = P(A_1) + P(A_2) + \text{--} + P(A_K)$$

Replacing A_K by $A_K \cup A_{K+1}$, we get

$$P[A_1 \cup A_2 \cup \text{—} \cup (A_K \cup A_{K+1})]$$

$$= P[(A_1 \cup A_2 \text{—} \cup A_K) \cup A_{K+1}]$$

$$= P(A_1 \cup A_2 \cup \text{—} \cup A_K) + P(A_{K+1})$$

$$= P(A_1) + P(A_2) + \text{————} + P(A_{K+1})$$

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which shows that statement is true for $n = K+1$

Hence by Mathematical Induction, it is true for all possitive integers 'say' 'n'. i-e

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) \quad \text{Required.}$$

10- If 'A' and 'B' are any two events then $P(A \cup B) \leq P(A) + P(B)$

If 'A' and 'B' are any two events in a sample space 'S', then we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad \text{--- (1)}$$

Since $0 \leq P(A \cap B) \leq 1$

i-e $P(A \cap B) \geq 0$

Thus if $P(A \cap B) = 0$, then (1) becomes

$$P(A \cup B) = P(A) + P(B) \quad \text{--- (a)}$$

If $P(A \cap B) > 0$, so (1) becomes

$$P(A \cup B) < P(A) + P(B) \quad \text{--- (b)}$$

Combining (a) & (b), we get

$$P(A \cup B) \leq P(A) + P(B) \quad \text{Required.}$$

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11- MULTIPLICATION LAW (Compound probability)

Statement:

If A and B are any two events belonging to a sample space S. Then

$$P(A \cap B) = P(A) \cdot P(B/A)$$
$$= P(B) \cdot P(A/B)$$

PROOF:

Let S be a sample space containing 'n' outcomes and A, B are any two events in S. Let n_1, n_2, n_3 are number of outcomes in A, B and $A \cap B$ respectivily.

Then $P(A) = \frac{n_1}{n}$, $P(B) = \frac{n_2}{n}$, $P(A \cap B) = \frac{n_3}{n}$

Now $P(A \cap B) = \frac{n_3}{n}$

$$= \frac{n_3/n_1}{n/n_1}$$

$$= \frac{n_3}{n_1} \cdot \frac{n_1}{n}$$

$$= \frac{n_3/n}{n_1/n} \cdot \frac{n_1}{n}$$

$$= \frac{P(A \cap B)}{P(A)} \cdot P(A)$$

$$= P(B/A) \cdot P(A)$$

Thus $P(A \cap B) = P(A) \cdot P(B/A)$

Similarly, we can prove

$$P(A \cap B) = P(B) \cdot P(A/B)$$

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Corollary: If A, B and C are any three events, then

$$P(A \cap B \cap C) = P(A) \cdot P(B/A) \cdot P(C/A \cap B)$$

PROOF:

$$P(A \cap B \cap C) = P(D \cap C) \quad \text{where } D = A \cap B$$

$$= P(D) \cdot P(C/D)$$

$$= P(A \cap B) \cdot P(C/A \cap B)$$

Hence $P(A \cap B \cap C) = P(A) \cdot P(B/A) \cdot P(C/A \cap B)$

Generalizing the results for n events

$$P(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n) = P(A_1) \cdot P(A_2/A_1) \cdot P(A_3/A_1 \cap A_2) \cdots P(A_n/A_1 \cap A_2 \cap \cdots \cap A_{n-1})$$

DEFINITION: Two events A & B are said to be independent if $P(A \cap B) = P(A) \cdot P(B)$.

12- If A and B are any two independent events in a sample space S. Then so is

(i) A and $\bar{B}$ are independent.

(ii) $\bar{A}$ and B is independent.

(iii) $\bar{A}$ and $\bar{B}$ are independent.

PROOF: Since A and B are independent

(i) So $P(A \cap B) = P(A) \cdot P(B)$ — ①

Also we know

$$P(A \cap \bar{B}) = P(A) - P(A \cap B)$$

$$= P(A) - P(A) \cdot P(B) \quad \text{using ①}$$

$$= P(A)[1 - P(B)]$$

So $P(A \cap \bar{B}) = P(A) \cdot P(\bar{B})$

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(ii) We know

$$P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

$$= P(B) - P(A) \cdot P(B) \quad \text{using ①}$$

$$= P(B)[1 - P(A)]$$

Hence

$$P(\bar{A} \cap B) = P(B)\, P(\bar{A})$$

i.e. $P(\bar{A} \cap B) = P(\bar{A}) \cdot P(B)$

(iii) By De Morgan's Law

$$\bar{A} \cap \bar{B} = \overline{(A \cup B)}$$

$$\Rightarrow P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B})$$

$$= 1 - P(A \cup B)$$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - P(A) - P(B) + P(A) \cdot P(B)$$

$$= P(\bar{A}) - P(B)[1 - P(A)]$$

$$= P(\bar{A}) - P(B)\, P(\bar{A})$$

$$= P(\bar{A})[1 - P(B)]$$

Hence $P(\bar{A} \cap \bar{B}) = P(\bar{A}) \cdot P(\bar{B})$

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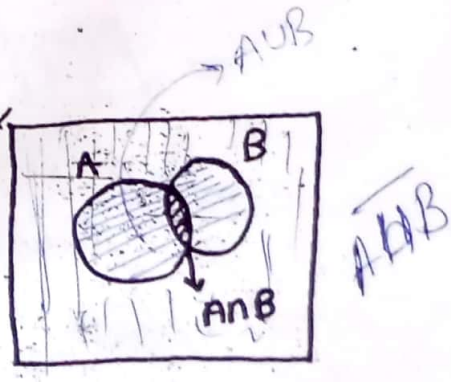
13- If 'A' and B are any two independent events, then show that

$$P(A \cup B) = 1 - P(\bar{A}) \cdot P(\bar{B})$$

Since A and B are independent

So $P(A \cap B) = P(A) \cdot P(B)$ —— ①



Also from Venn diagram

$A \cup B$ and $\overline{A \cup B}$ are exhaustive events.

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So $(A\cup B)\cup(\overline{A\cup B}) = S$

$\Rightarrow P[(A\cup B)\cup(\overline{A\cup B})] = P(S)$

$\Rightarrow P(A\cup B) + P(\overline{A\cup B}) = 1$

$P(A\cup B) = 1 - P(\overline{A\cup B})$

$= 1 - P(\bar{A}\cap\bar{B})$ $\because \overline{A\cup B} = \bar{A}\cap\bar{B}$

$= 1 - P(\bar{A})\cdot P(\bar{B})$ $\because \bar{A}$ & $\bar{B}$ are independent.

Thus

$P(A\cup B) = 1 - P(\bar{A})\cdot P(\bar{B})$

14- If A, B and C are independent events, then show that $A\cup B$ and C are also independent.

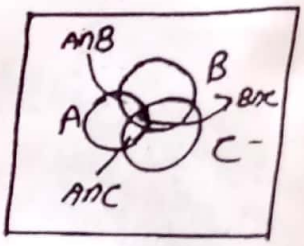
We are to show that

$P[(A\cup B)\cap C] = P(A\cup B)\cdot P(C)$

Now

$P[(A\cup B)\cap C] = \cancel{P\,AAC}$

$= P[(A\cap C)\cup(B\cap C)]$



$= P(A\cap C) + P(B\cap C) - P[(A\cap C)\cap(B\cap C)]$ using venn diagram

$= P(A\cap C) + P(B\cap C) - P(A\cap B\cap C)$

$= P(A)\cdot P(C) + P(B)\cdot P(C) - P(A)\cdot P(B)\cdot P(C)$ $\because$ A, B & C are independent

$= P(C)[P(A) + P(B) - P(A)P(B)]$

$= P(C)[P(A) + P(B) - P(A\cap B)]$ $\because$ A & B are independent.

$= P(C)\,P(A\cup B)$

$= P(A\cup B)\cdot P(C)$

Hence $P[(A\cup B)\cap C] = P(A\cup B)\cdot P(C)$

That is $A\cup B$ and C are independent.

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(15) If A is independent of B, $B \cap C$ and $B \cup C$ Then show that A is independent of C.

We are to show that

$$P(A \cap C) = P(A) \cdot P(C)$$

We are given

$$P(A \cap B) = P(A) \cdot P(B) \quad \text{--- (1)}$$

$$P[A \cap (B \cap C)] = P(A) \cdot P(B \cap C) \quad \text{--- (2)}$$

$$P[A \cap (B \cup C)] = P(A) \cdot P(B \cup C) \quad \text{--- (3)}$$

Consider

$$P[A \cap (B \cup C)] = P(A) \cdot P(B \cup C)$$

$$= P(A) \cdot [P(B) + P(C) - P(B \cap C)]$$

$$= P(A) \cdot P(B) + P(A) \cdot P(C) - P(A) \cdot P(B \cap C)$$

$$= P(A \cap B) + P(A) \cdot P(C) - P[A \cap (B \cap C)] \quad \text{using (1), (2)}$$

$$\Rightarrow P(A \cap B \cap C) = P(A \cap B) + P(A) \cdot P(C) - P[A \cap (B \cup C)]$$

$$\Rightarrow P[A \cap (B \cup C)] = P(A \cap B) + P(A) \cdot P(C) - P(A \cap B \cap C) \quad \text{--- (4)}$$

Also consider

$$P[A \cap (B \cup C)] = P[(A \cap B) \cup (A \cap C)]$$

$$= P(A \cap B) + P(A \cap C) - P[(A \cap B) \cap (A \cap C)]$$

$$P[A \cap (B \cup C)] = P(A \cap B) + P(A \cap C) - P(A \cap B \cap C) \quad \text{--- (5)}$$

Equating from (4) & (5) we get

$$P(A \cap B) + P(A \cap C) - P(A \cap B \cap C) = P(A \cap B) + P(A) \cdot P(C) - P(A \cap B \cap C)$$

$$\Rightarrow P(A \cap C) = P(A) \cdot P(C)$$

Hence A and C are independent.

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16- If A, B and C are pairwise independent events in a sample space S and A is independent of $B \cup C$. Then prove that A, B and C are mutually independent.

Since A, B and C are pair-wise independent,

$$\left.\begin{aligned} P(A \cap B) &= P(A) \cdot P(B) \\ P(A \cap C) &= P(A) \cdot P(C) \\ P(B \cap C) &= P(B) \cdot P(C) \end{aligned}\right\} \text{—— (1)}$$

Also we are given

$$P[A \cap (B \cup C)] = P(A) \cdot P(B \cup C) \text{ —— (2)}$$

Now

$$P[A \cap (B \cup C)] = P[(A \cap B) \cup (A \cap C)]$$

$$= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$$

$$\Rightarrow P(A \cap B \cap C) = P(A \cap B) + P(A \cap C) - P[A \cap (B \cup C)]$$

$$= P(A) \cdot P(B) + P(A) \cdot P(C) + P(A) \cdot P(B \cup C) \quad \text{using (1) \& (2)}$$

$$= P(A) \cdot P(B) + P(A) \cdot P(C) - P(A)[P(B) + P(C) - P(B \cap C)]$$

$$= P(A) \cdot P(B) + P(A) \cdot P(C) - P(A)[P(B) + P(C) - P(B) \cdot P(C)] \quad \text{using (1)}$$

$$= \cancel{P(A) \cdot P(B)} + \cancel{P(A) \cdot P(C)} - \cancel{P(A) \cdot P(B)} - \cancel{P(A) \cdot P(C)} + P(A) \cdot P(B) \cdot P(C)$$

Here

$$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$$

That is A, B and C are mutually independent.

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17- Find the expression for the events A, B and C when

(a) (i) only A occur (ii) None occur

(iii) Atleast two occur

(b) Prove for any event $0 \le P(A/B) \le 1$, where $P(B) > 0$, B being another event.

(a) (i) Only A occur means

For any $x \in A$, but $x \notin B$ and $x \notin C$

$\Rightarrow x \in A$ and $x \in \bar{B}$ and $x \in \bar{C}$

$\Rightarrow x \in A \cap \bar{B} \cap \bar{C}$

Where $\bar{B}$ and $\bar{C}$ are complements of B and C.

Hence only A occur means $A \cap \bar{B} \cap \bar{C}$.

(ii) None occur.

For any $x \notin A$ and $x \notin B$ and $x \notin C$

$\Rightarrow x \in \bar{A}$ and $x \in \bar{B}$ and $x \in \bar{C}$

$\Rightarrow x \in \bar{A} \cap \bar{B} \cap \bar{C}$

$\Rightarrow x \in \overline{A \cup B \cup C}$

Then none of A, B and C occur mean the event $\overline{A \cup B \cup C}$

(iii) Atleast two occur.

For any $x \in A$, $x \in B$, $x \notin C$
or $x \in A$, $x \notin B$, $x \in C$
or $x \in \bar{A}$, $x \in B$, $x \in C$
or $x \in A$, $x \in B$, $x \in C$ — ①

① ⇒ $x \in A$, $x \in B$, $x \in \bar{C}$
or $x \in A$, $x \in \bar{B}$, $x \in C$
or $x \in \bar{A}$, $x \in B$, $x \in C$
or $x \in A$, $x \in B$, $x \in C$ — ②

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$$\Rightarrow \quad x \in A \cap B \cap \bar{C}$$
$$\text{or } x \in A \cap \bar{B} \cap C$$
$$\text{or } x \in \bar{A} \cap B \cap C$$
$$\text{or } x \in A \cap B \cap C$$

$$\Rightarrow \quad x \in (A \cap B \cap \bar{C}) \cup (A \cap \bar{B} \cap C) \cup (\bar{A} \cap B \cap C) \cup (A \cap B \cap C)$$

Hence atleast two occur means the event

$$(A \cap B \cap \bar{C}) \cup (A \cap \bar{B} \cap C) \cup (\bar{A} \cap B \cap C) \cup (A \cap B \cap C)$$

(b) From last results, we know (or by using Venn Diagram)

$$B = (A \cap B) \cup (\bar{A} \cap B)$$

Moreover $A \cap B$ and $\bar{A} \cap B$ are mutually exclusive events. Thus

$$P(B) = P(A \cap B) + P(\bar{A} \cap B)$$

$$\Rightarrow P(B) \geq P(A \cap B) \quad \because P(\bar{A} \cap B) \geq 0$$

$$\Rightarrow \frac{P(A \cap B)}{P(B)} \leq 1 \quad , P(B) > 0 \text{ given}$$

$$\Rightarrow P(A/B) \leq 1 \text{ ——— } (1)$$

Also by definition of probability

$$P(A/B) \geq 0 \text{ ——— } (2)$$

Combining from (1) & (2), we get

$$0 \leq P(A/B) \leq 1 \quad \text{Required}$$

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18- If A and B are any two events and $P(B) \neq 1$, then prove that

$$P(A/\bar{B}) = \frac{P(A) - P(A \cap B)}{1 - P(B)}$$

and deduce that

$$P(A \cap B) \geq P(A) + P(B) - 1$$

From conditional probability, we know that

$$P(A/\bar{B}) = \frac{P(A \cap \bar{B})}{P(\bar{B})} \quad \text{--- (1)}$$

Also, we know that

$$P(A \cap \bar{B}) = P(A) - P(A \cap B)$$

& $P(\bar{B}) = 1 - P(B)$

Putting values in (1), we get

$$P(A/\bar{B}) = \frac{P(A) - P(A \cap B)}{1 - P(B)}$$

For deduction, we have

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$$0 \leq P(A/\bar{B}) \leq 1$$

$$\Rightarrow P(A/\bar{B}) \leq 1$$

$$\Rightarrow \frac{P(A) - P(A \cap B)}{1 - P(B)} \leq 1$$

$$\Rightarrow P(A) - P(A \cap B) \leq 1 - P(B)$$

$$\Rightarrow P(A) + P(B) - 1 \leq P(A \cap B)$$

$$\Rightarrow P(A \cap B) \geq P(A) + P(B) - 1$$

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19- If A and B are mutually-exclusive events and $P(A \cup B) \neq 0$, then

$$P(A / A \cup B) = \frac{P(A)}{P(A) + P(B)}$$

Since A and B are mutually-exclusive events, so

$$P(A \cup B) = P(A) + P(B) \quad \text{—— (1)}$$

Also $A \cap B = \Phi$

Moreover

$$A \cap (A \cup B) = (A \cap A) \cup (A \cap B)$$
$$= A \cup \Phi$$
$$= A$$

$$\Rightarrow P[A \cap (A \cup B)] = P(A) \quad \text{—— (2)}$$

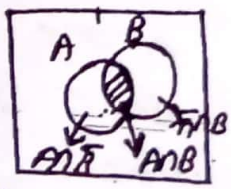
Now $P(A / A \cup B) = \dfrac{P[A \cap (A \cup B)]}{P(A \cup B)}$ By Conditional Probabil

Hence $P(A / A \cup B) = \dfrac{P(A)}{P(A) + P(B)}$ using (1) & (2).

20- For any two dependent events A and B show that

$$P(B) = P(A) \cdot P(B/A) + P(\bar{A}) \cdot P(B/\bar{A})$$

From Venn diagram, we have



$$B = (A \cap B) \cup (\bar{A} \cap B)$$

$$P(B) = P(A \cap B) + P(\bar{A} \cap B)$$ as $A \cap B$ & $\bar{A} \cap B$ are mutually-exclusive

$$\Rightarrow P(B) = P(A) \cdot P(B/A) + P(\bar{A}) \cdot P(B/\bar{A})$$

Required.

Corollary:
(ii)

$$P(A) = P(A) \cdot P(B/A) + P(\bar{B}) \cdot P(A/\bar{B})$$

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21- Prove that

(i) $A = AB \cup A\bar{B}$

(ii) $(A \cup B)C = AC \cup BC$

(i) Consider

$$AB \cup A\bar{B} = (A \cap B) \cup (A \cap \bar{B})$$
$$= A \cap (B \cup \bar{B})$$
$$= A \cap S$$
$$= A$$

Hence $A = AB \cap A\bar{B}$

(ii) $(A \cup B)C = AC \cup BC$

Let $x \in (A \cup B)C$

$\Rightarrow x \in A \cup B$ and $x \in C$

$\Rightarrow x \in A$ or $x \in B$ and $x \in C$

If $x \in A$ and $x \notin B$, then $x \in A$ and $x \in C$

That is $x \in A \cap C$

$\Rightarrow x \in AC$ —— ①

If $x \notin A$ and $x \in B$, then $x \in B$ and $x \in C$

That is $x \in B \cap C$

$\Rightarrow x \in BC$ —— ②

From ① & ②, we get

$x \in AC$ or $x \in BC$

Hence $x \in AC \cup BC$

Since 'x' was arbitrary

Thus, $(A \cup B)C \subseteq AC \cup BC$ —— (3)

Conversely, let

$x \in AC \cup BC$

$\Rightarrow x \in AC$ or $x \in BC$

$\Rightarrow x \in A \cap C$ or $x \in B \cap C$

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$\Rightarrow x \in A$ and $x \in C$

or $x \in B$ and $x \in C$

Thus in both cases

$x \in A$ or $x \in B$ and $x \in C$

$\Rightarrow x \in A \cup B$ and $x \in C$

$\Rightarrow x \in (A \cup B) C$

Hence $AC \cup BC \subseteq (A \cup B) C$ — (4)

From (3) & (4), we get

$$(A \cup B) C = AC \cup BC$$

22- Prove the followings.

(i) $A = AB \cup A\bar{B}$

(ii) $A \cup B = A\bar{B} \cup \bar{A}B \cup AB$

(i) Already

(ii) $A\bar{B} \cup \bar{A}B \cup AB = (A \cap \bar{B}) \cup (\bar{A} \cap B) \cup (A \cap B)$

$= (A \cap \bar{B}) \cup [B \cap (\bar{A} \cup A)]$

$= (A \cap \bar{B}) \cup (B \cap S)$

$= (A \cap \bar{B}) \cup B$

$= \cancel{AB}\ (A \cup B) \cap (\bar{B} \cup B)$

$= (A \cup B) \cap S$

$= A \cup B$

Hence $A \cup B = A\bar{B} \cup \bar{A}B \cup AB$.

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23- Find simple expression for

(i) $(A\cup B)(B\cup C)$

(ii) $(A\cup B)(A\cup \bar{B})$

(iii) $(A\cup B)(\bar{A}\cup B)(A\cup \bar{B})$

(i) $(A\cup B)(B\cup C)$

$= (A\cup B)\cap(B\cup C)$

$= (B\cup A)\cap(B\cup C)$

$= B\cup(A\cap C)$

$= B\cup AC$

$= AC\cup B$

Hence $(A\cup B)(B\cup C) = AC\cup B$

(ii) $(A\cup B)(A\cup \bar{B})$

$= (A\cup B)\cap(A\cup \bar{B})$

$= A\cup(B\cap \bar{B})$

$= A\cup \Phi$

$= A$

Hence $(A\cup B)(A\cup \bar{B}) = A$

(iii) $(A\cup B)(\bar{A}\cup B)(A\cup \bar{B})$

$= (A\cup B)\cap(\bar{A}\cup B)\cap(A\cup \bar{B})$

$= \{(A\cup B)\cap(\bar{A}\cup B)\}\cap(A\cup \bar{B})$

$= \{B\cup(A\cap \bar{A})\}\cap(A\cup \bar{B})$

$= (B\cup \Phi)\cap(A\cup \bar{B})$

$= B\cap(A\cup \bar{B})$

$= (B\cap A)\cup(B\cap \bar{B})$

$= A\cap B\cup \Phi = A\cap B = AB$

Hence $(A\cup B)(\bar{A}\cup B)(A\cup \bar{B}) = AB$

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24- For any three events A, B and C in a a sample space S. Prove that

$$P(A\cap\bar{B}/C) + P(A\cap B/C) = P(A/C)$$

provided $C \neq \Phi$

From conditional probability, we have

$$P(A\cap\bar{B}/C) + P(A\cap B/C)$$

$$= \frac{P[(A\cap\bar{B})\cap C]}{P(C)} + \frac{P[(A\cap B)\cap C]}{P(C)}$$

$$= \frac{P[(A\cap\bar{B})\cap C] + P[(A\cap B)\cap C]}{P(C)} \quad \text{--- (1)}$$

From Venn diagram, we can easily see that $A\cap\bar{B}\cap C$ and $A\cap B\cap C$ are mutually exclusive events. Thus equation (1) becomes.

$$= \frac{P[\{(A\cap\bar{B})\cap C\}\cup\{(A\cap B)\cap C\}]}{P(C)}$$

$$= \frac{P[\{(A\cap C)\cap\bar{B}\}\cup\{(A\cap C)\cap B\}]}{P(C)}$$

$$= \frac{P[(A\cap C)\cap(\bar{B}\cup B)]}{P(C)}$$

$$= \frac{P[(A\cap C)\cap S]}{P(C)} = \frac{P(A\cap C)}{P(C)}$$

$= P(A/C)$, Hence

$$P(A\cap\bar{B}/C) + P(A\cap B/C) = P(A/C)$$

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Corollary: Prove that

$$P(\bar{A}\cap B/C) + P(A\cap B/C) = P(B/C), \quad C \neq \Phi$$

85- Prove that

(i) $P(AB) + P(\bar{A}B) + P(A\bar{B}) + P(\bar{A}\bar{B}) = 1$

(ii) $P(\bar{A}\bar{B}) = 1 + P(AB) - P(A) - P(B)$

(i) $P(AB) + P(\bar{A}B) + P(A\bar{B}) + P(\bar{A}\bar{B})$

$= P(A\cap B) + P(\bar{A}\cap B) + P(A\cap\bar{B}) + P(\bar{A}\cap\bar{B})$

$= P[(A\cap B)\cup(\bar{A}\cap B)\cup(A\cap\bar{B})] + P(\bar{A}\cap\bar{B})$ because $A\cap B$, $\bar{A}\cap B$ and $A\cap\bar{B}$ are mutually exclusive.

$= P(A\cup B) + P(\bar{A}\cap\bar{B})$

$= P(A\cup B) + P(\overline{A\cup B})$

$= P[(A\cup B)\cup(\overline{A\cup B})]$ since $A\cup B$ and its complement are mutually exclusive.

$= P(S)$

$= 1$

Hence $P(AB) + P(A\bar{B}) + P(\bar{A}B) + P(\bar{A}\bar{B}) = 1$

(ii) $P(\bar{A}\bar{B}) = P(\bar{A}\cap\bar{B})$

$= P(\overline{A\cup B})$

$= 1 - P(A\cup B)$

$= 1 - [P(A) + P(B) - P(A\cap B)]$

$= 1 - P(A) - P(B) + P(A\cap B)$

$= 1 + P(AB) - P(A) - P(B)$

Hence $P(\bar{A}\bar{B}) = 1 + P(AB) - P(A) - P(B)$

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26- Prove the following cases, assuming that in each case the conditional probability is defined as

(i) For a sure event S,

$$P(S/B) = 1, \quad P(B) \neq 0$$

(ii) $P(A_1 \cup A_2 / B) = P(A_1/B) + P(A_2/B) - P(A_1 \cap A_2/B)$

But if A_1 and A_2 are mutually-exclusive and $P(B) > 0$

$$P(A_1 \cup A_2/B) = P(A_1/B) + P(A_2/B)$$

or in general if $A_1, A_2, \ldots, A_n$ is a sequence of n mutually exclusive events, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n/B) = P(A_1/B) + P(A_2/B) + \cdots + P(A_n/B)$$

that is $P\left(\bigcup_{i=1}^{n} A_i/B\right) = \sum_{i=1}^{n} P(A_i/B)$

(i) $P(S/B) = \dfrac{P(S \cap B)}{P(B)}$

$= \dfrac{P(B)}{P(B)}$ since $S \cap B = B$

$= 1$

Hence, $P(S/B) = 1$

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(ii) $P(A_1 \cup A_2/B) = \dfrac{P[(A_1 \cup A_2) \cap B]}{P(B)}$

$= \dfrac{P[(A_1 \cap B) \cup (A_2 \cap B)]}{P(B)}$

$= \dfrac{P(A_1 \cap B) + P(A_2 \cap B) - P[(A_1 \cap B) \cap (A_2 \cap B)]}{P(B)}$

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that is

$$P(A_1 \cup A_2/B) = \frac{P(A_1 \cap B) + P(A_2 \cap B) - P[(A_1 \cap A_2) \cap B]}{P(B)}$$

$$= \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)} - \frac{P[(A_1 \cap A_2) \cap B]}{P(B)}$$

$$= \frac{P(B) \cdot P(A_1/B)}{P(B)} + \frac{P(B) \cdot P(A_2/B)}{P(B)} - \frac{P(B) \cdot P(A_1 \cap A_2/B)}{P(B)}$$

Hence

$$P(A_1 \cup A_2/B) = P(A_1/B) + P(A_2/B) - P(A_1 \cap A_2/B)$$

In case A_1 and A_2 are mutually exclusive events, then $A_1 \cap B$ and $A_2 \cap B$ are also mutually exclusive.

$$\cancel{P(A_1 \cup A_2/B)} =$$

$$P(A_1 \cup A_2/B) = \frac{P[(A_1 \cup A_2) \cap B]}{P(B)}$$

$$= \frac{P[(A_1 \cap B) \cup (A_2 \cap B)]}{P(B)}$$

$$= \frac{P(A_1 \cap B) + P(A_2 \cap B)}{P(B)}$$

$$= \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)}$$

$$= P(A_1/B) + P(A_2/B)$$

Again if $A_1, A_2, \ldots, A_n$ are mutually exclusive events, then $A_1 \cap B, A_2 \cap B, \ldots, A_n \cap B$ are also mutually exclusive events.

Thus $P(A_1 \cup A_2 \cup \cdots \cup A_n/B) = \dfrac{P[(A_1 \cup A_2 \cup \cdots \cup A_n) \cap B]}{P(B)}$

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That is

$$P[A_1 \cup A_2 \cup \cdots \cup A_n / B] = \frac{P[(A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B)]}{P(B)}$$

$$= \frac{P(A_1 \cap B) + P(A_1 \cap B) + \cdots + P(A_n \cap B)}{P(B)}$$

$$= \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)} + \cdots + \frac{P(A_n \cap B)}{P(B)}$$

$$= P(A_1/B) + P(A_2/B) + \cdots + P(A_n/B)$$

Hence

$$P\left(\bigcup_{i=1}^{n} A_i / B\right) = \sum_{i=1}^{n} P(A_i/B)$$

27- Prove that

(i) $0 \le P(AB) \le P(A \cup B) \le P(A) + P(B) \le 1$

(ii) $2P(AB) + P(A\bar{B} \cup \bar{A}B) = P(A) + P(B)$

(iii) $P(B) = 1 \Rightarrow P(A) = P(A/B)$

(i) we know that $P(\Phi) = 0$; $P(S) = 1$

$$P(A \cap B) = P(A) \cdot P(B/A)$$

$$\Rightarrow P(A \cap B) \geq 0 \quad \because P(A) \geq 0 \text{ ; } P(B/A) \geq 0$$

So $0 \le P(A \cap B)$ — ①

Moreover $\frac{P(A \cap B)}{P(A)} = P(B/A) \le 1$

$$\Rightarrow P(A \cap B) \le P(A) \quad \text{— ②}$$

Also, we know

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow P(A \cup B) \geq P(A) \quad \because P(B) - P(A \cap B) \geq 0.$$

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$$P(A) \le P(A \cup B) \quad \text{--- (3)}$$

Also we have proved

$$P(A \cup B) \le P(A) + P(B) \quad \text{--- (4)}$$

Combining (1), (2), (3), (4), we get

$$0 \le P(A \cap B) \le P(A) \le P(A \cup B) \le P(A) + P(B) \le 1$$

Hence

$$0 \le P(AB) \le P(A \cup B) \le P(A) + P(B) \le 1$$

(ii) Already

(iii) $P(B) = 1 \Rightarrow B = S$

$\because$ only $P(S) = 1$

Now

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap S)}{P(B)}$$

$$= \frac{P(A)}{1} = P(A)$$

Hence $P(A) = P(A/B)$.

38- If 'A' and B are independent,

Then (i) $P(A/B) = P(A)$

(ii) $P(B/A) = P(B)$

we have

(i) $$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{P(A) \cdot P(B)}{P(B)} \quad \because A \text{ \& } B \text{ are independent.}$$

Hence $P(A/B) = P(A)$

Similarly, we can prove second.

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If $A \cap B = \Phi$, then show that

$$P(A) \leq P(\bar{B})$$

we know

$$A = (A \cap B) \cup (A \cap \bar{B})$$

$$A = \Phi \cup (A \cap \bar{B})$$

$$A = (A \cap \bar{B})$$

$$\Rightarrow A \subseteq \bar{B}$$

$$\Rightarrow P(A) \leq P(\bar{B}).$$

For any three events A, B and C, prove that

$$P(A \cup B \cup C) = 1 - P(\bar{A}) \cdot P(\bar{B}) \cdot P(\bar{C})$$

we know

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$$

$$= P(A) - P(A) \cdot P(C) + P(B) - P(B) \cdot P(C) - P(B) \cdot P(A) + P(A) \cdot P(B) P(C) + P(C)$$

$$= P(A) P(\bar{C}) + P(B) P(\bar{C}) - P(B) \cdot P(A) \cdot P(\bar{C}) + P(C)$$

$$= P(\bar{C}) \left[P(A) + P(B) - P(A) \cdot P(B) \right] + 1 - P(\bar{C})$$

$$= P(\bar{C}) P(A \cup B) - P(\bar{C}) + 1$$

$$= 1 - P(\bar{C}) \left[1 - P(A \cup B) \right]$$

$$= 1 - P(\bar{C}) P(\overline{A \cup B}) = 1 - P(\bar{A} \cap \bar{B}) \cdot P(\bar{C})$$

hence

$$P(A \cup B \cup C) = 1 - P(\bar{A}) \cdot P(\bar{B}) \cdot P(\bar{C})$$

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Bayes THEOREM:

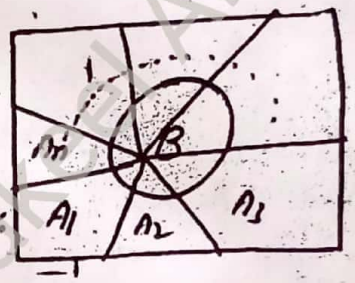
STATEMENT: 2012, 2015.

Let S be a sample space and $A_1, A_2, \ldots, A_n$ form the partition of sample space S. That is events A_i $1 \le i \le n$ are exhaustive events. If B is any other events of S such that it can occur only if one of the A_i occur. Then for any i

$$P(A_i/B) = \frac{P(A_i)\,P(B/A_i)}{\sum_i P(A_i)\,P(B/A_i)}$$

PROOF:

Let S be a sample space in which A_i are mutually exhaustive events and any B occur only if one of the A_i occur.



By Multiplication law of probability, we have

$$P(B \cap A_i) = P(A_i) \cdot P(B/A_i)$$
$$= P(B) \cdot P(A_i/B)$$

That is

$$P(B) \cdot P(A_i/B) = P(A_i) \cdot P(B/A_i)$$
$$P(A_i/B) = \frac{1}{P(B)}\left[P(A_i) \cdot P(B/A_i)\right] \quad \text{—①}$$

We have

$$B = S \cap B$$
$$= (A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n) \cap B$$
$$B = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B)$$
$$\Rightarrow P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \cdots + P(A_n \cap B) \quad \because A_i \cap B \text{ are M.E.}$$
$$P(B) = P(A_1) \cdot P(B/A_1) + P(A_2)\,P(B/A_2) + \cdots + P(A_n) \cdot P(B/A_n)$$
$$P(B) = \sum_i P(A_i) \cdot P(B/A_i) \quad \text{put in ①}$$

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PROBLEMS

1- 2010 Three urns of the same appearance have the following composition of balls

	I	II	III
White	1	2	2
Black	2	1	2

one of the urn is selected and a ball is taken. It turn out to be white. What is the probability that it came from urn three.

SOLUTION: Let A_1, A_2, A_3 be the events, stand, for the selection of urns I, II and III resp.

Then $P(A_1) = P(A_2) = P(A_3) = 1/3$

Let B be stand for white ball after selection.

$P(B/A_1) = 1/3$. $P(B/A_2) = 2/3$, $P(B/A_3) = 2/4 = \frac{1}{2}$

It came from urn three if

$$P(A_3/B) = \frac{P(A_3) \cdot P(B/A_3)}{\sum_{i=1}^{3} P(A_i)\, P(B/A_i)}$$

$$= \frac{1/3 \cdot 1/2}{1/3 \cdot 1/3 + 1/3 \cdot 2/3 + 1/3 \cdot 1/2} = 1/3$$

$$P(A_3/B) = 1/3$$

A/2015 A/2018 2- There are three coins, identical in appearance, one of which is ideal and other two are biased with probabilities 1/3 and 2/3 respectivily for a head occur. One coin is taken at random and twicely tossed. If a head appear both the times, what is the probability that the ideal coin was choosen.

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Solution: Let A_1, A_2, A_3 are three coin of which first is ideal.

Then $P(A_1) = P(A_2) = P(A_3) = 1/3$

Let B stand for the occurance of head

$P(B/A_1) = \frac{1}{2} \cdot \frac{1}{2} = 1/4$ ∵ First is good and coin is tossed twicely.

$P(B/A_2) = \frac{1}{3} \cdot \frac{1}{3} = 1/9$

$P(B/A_3) = 2/3 \cdot 2/3 = 4/9$

$$P(A_1/B) = \frac{P(A_1) \cdot P(B/A_1)}{\sum_{i=1}^{3} P(A_i) P(B/A_i)} = \frac{1/3 \cdot 1/4}{1/3 \cdot 1/4 + 1/3 \cdot 1/9 + 1/3 \cdot 4/9}$$

$= 9/29$ Ans.

3- 2010, 2011 In a bolt factory, machines A, B, C manufacture 25%, 35% and 40% respectively of the total. Of their output 5%, 4% and 2% are defective bolts. A bolt is drawn at random and found to be defective. What are the probabilities that it was manufactured by machines A, B or C.

SOLUTION: we are given

$P(A) = 0.25$, $P(B) = 0.35$, $P(C) = 0.4$

Let K stands for drawing a defective

$P(K/A) = 0.05$, $P(K/B) = 0.04$, $P(K/C) = 0.02$

(i) $$P(A/K) = \frac{P(A) \cdot P(K/A)}{P(A) \cdot P(K/A) + P(B) P(K/B) + P(C) \cdot P(K/C)}$$

$$= \frac{(0.25)(0.05)}{(0.25)(0.05) + (0.35)(0.04) + (0.4)(0.02)}$$

$P(A/K) = 0.3630$

Similarly $P(B/K) = 0.406$, $P(C/K) = 0.232$

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4- In a certain college, the geographical distribution of male students is as follows. 50% come from east, 30% from mid west and 20% come from the far west. The following proportion of the male students wear ties 80% of the Eeastern, 60% of the mid western and 40% of far western. What is the probability that a student who wear a tie comes from the east.

Solution:

Let A_1, A_2, A_3 stands for Eeast, Mid west and for far western male students.

Then $P(A_1) = 0.50$, $P(A_2) = 0.30$, $P(A_3) = 0.20$

Let E stands for their wearing tie.

$P(E/A_1) = 0.8$, $P(E/A_2) = 0.6$, $P(E/C) = 0.4$

$$P(A_1/E) = \frac{P(A_1) \cdot P(E/A_1)}{\sum_{i=1}^{3} P(A_i) P(E/A_i)}$$

$$= \frac{(0.5)(0.8)}{(0.5)(0.8) + (0.3)(0.6) + (0.2)(0.4)}$$

$$= 0.61 \text{ Ans.}$$

5- Three bakers A, B and C bake a special kind of cake and with respective probabilities 0.02, 0.03 and 0.05 it fails to rise. In the restaurants where they work, A bake 50%, B bake 30% and C bake 20%. What proportion of failure is caused by A.

SOLUTION: $P(A) = 0.5$, $P(B) = 0.3$, $P(C) = 0.20$

Let $\bar{B}$ denote that cake fails to rise

$P(\bar{B}/A) = 0.02$, $P(\bar{B}/B) = 0.03$, $P(\bar{B}/C) = 0.0$

$$P(A/\bar{B}) = \frac{P(\bar{B}/A) \cdot P(A)}{P(A) \cdot P(\bar{B}/A) + P(B) \cdot P(\bar{B}/B) + P(C)\, P(\bar{B}/C)}$$

$$= \frac{(0.50)(0.02)}{(0.50)(0.02) + (0.3)(0.03) + (0.2)(0.05)} = 0.3448.$$

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6- In a college 4% of the men and 1% of the women are taller than 6 feet. Furthermore 60% of the students are women. Now if a student is selected at random and is taller than 6 feet what the probability that student is women.

Solution: Let W stand for women and M stand for men,

$$P(W) = 0.60 \quad , \quad P(M) = 0.40$$

Let B stand for taller than 6 feet if it is men and A stand for taller than 6 feet if it is women.

$$P(A/W) = 0.01, \qquad P(B/M) = 0.04$$

Thus required is

$$P(W/A) = \frac{P(W) \cdot P(A/W)}{P(W) \cdot P(A/W) + P(M) \cdot P(B/M)}$$

$$= 0.2727$$

7- Three members are nominated for the post of president. The probabilities of their selection are 0.3, 0.5 and 0.2 respectivily. Should the first be selected, the probability of an increase in membership fee is 0.8 and 0.1, 0.4 for second and third respectivily. Find that probability that third man is eleeted for president.

Solution: Let A_1, A_2, A_3 stands for the ~~selection of~~ first, second and third man

$$P(A_1) = 0.3, \quad P(A_2) = 0.5, \quad P(A_3) = 0.2$$

Let E for the increase in member ship fee.

$$P(E/A_1) = 0.8, \quad P(E/A_2) = 0.1, \quad P(E/A_3) = 0.4$$

$$P(A_3/E) = \frac{P(A_3) \cdot P(E/A_3)}{\sum_{i=1}^{3} P(A_i) P(E/A_i)} = 0.2162$$

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10- Three sisters S_1, S_2 and S_3 wash the dishes. They give 40%, 30% and 30% time respectively to wash. The probabilities of a dish is broken by them are 2%, 3% and 2% respectively, when they are on job. One night the parents do not know who is washing the dishes but hear the break. Find probability that S_1 was washing the dish.

SOLUTION: $P(S_1) = 0.4$, $P(S_2) = 0.3$, $P(S_3) = 0.3$

Let E stand for breaking the dish

$P(E/S_1) = 0.02$, $P(E/S_2) = 0.03$, $P(E/S_3) = 0.02$

$$P(S_1/E) = \frac{P(S_1).\ P(E/S_1)}{\sum_{i=1}^{3} P(S_i)\, P(E/S_i)}$$

$$= 0.3478$$

11- We have three urns

U_1 : one white, two black

U_2 : 2 W 3 B

U_3 : 2 W 2 B.

An urn is selected at random and ~~and~~ a ball drawn from the urn which is found to be white. Find probability of getting another white ball when first is not replaced.

SOLUTION: let A_i, $i = 1, 2, 3$ stands for the selection of urns.

$P(A_i) = 1/3$, $i = 1, 2, 3$.

Let E denote for coming white.

$P(E/A_1) = 1/3$, $P(E/A_2) = 2/5$, $P(E/A_3) = 1/2$

Let F stand for another white.

$P(F/A_1 \cap E) = 0$, $P(F/E \cap A_2) = 1/4$, $P(F/E \cap A_3) =$

$$P(F/E) = \frac{\sum P(A_i)\, P(E/A_i)\, P(F/A_i \cap E)}{\sum P(A_i)\, P(E/A_i)} = 8/37$$

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12- An urn contain 5 white and 5 black balls. From this urn five balls are transferred to another urn which was empty. A ball is drawn from second urn and is found to be white. Find the probability of another white ball ~~when~~ from second urn when first is not replaced.

SOLUTION:

Let B_i, $i = 1, 2, 3, 4, 5, 6$ be the events that stand for replacing the ball in second urn and this could be happen in such a way

Events	White	Black
B_1	0	5
B_2	1	4
B_3	2	3
B_4	3	2
B_5	4	1
B_6	5	0

Then $P(B_1) = \binom{5}{0}\binom{5}{5} / \binom{10}{5} = 1/252$

$P(B_2) = \binom{5}{1}\binom{5}{4} / \binom{10}{5} = 25/252$

$P(B_3) = \binom{5}{2}\binom{5}{3} / \binom{10}{5} = 100/252$

$P(B_4) = \binom{5}{3}\binom{5}{2} / \binom{10}{5} = 100/252$

$P(B_5) = \binom{5}{4}\binom{5}{1} / \binom{10}{5} = 25/252$

$P(B_6) = \binom{5}{5}\binom{5}{0} / \binom{10}{5} = 1/252$

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Let A denote the event that drawn ball from U_2 is white.

$P(A/B_1) = 0$, $P(A/B_2) = 1/5$, $P(A/B_3) = 2/5$

$P(A/B_4) = 3/5$, $P(A/B_5) = 4/5$, $P(A/B_6) = 5/5 = 1$

Let C denote the event that another taken is white when first is not replaced.

$P(C/A \cap B_1) = 0$, $P(C/A \cap B_2) = 0$, $P(C/A \cap B_3) = 1/4$

$P(C/A \cap B_4) = 1/2$, $P(C/A \cap B_5) = 3/4$, $P(C/A \cap B_6) = 4/4 = 1$

$$P(C/A) = \frac{\sum_{i=1}^{6} P(B_i) \cdot P(A/B_i) \cdot P(C/A \cap B_i)}{\sum_{i=1}^{6} P(B_i) \cdot P(A/B_i)} = 4/9$$

13- An urn contain 5 white and 5 black balls. Four balls are drawn from this urn to another urn which was empty. From this second, now, a ball is taken and is found to be white. what is the probability of drawing a white ball at the next drawn when the first drawn white ball is not replaced.

Ans. 4/9.

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Random Variable.

A numerical quantity whose value is determined by the outcome of a random experiment, is called a random variable. Mathematically, we assign a single real number to each outcome of a sample space, and hence we state that a random variable is a real-valued function defined on a sample space.

A random variable is also called a chance variable, or simply a variate and is abbreviated as r.v.

The random variables are usually denoted by capital letters such as X, Y, Z; while the values taken by them are represented by the corresponding small letters such as x, y, z.

It is to be noted that more than one random variable can be defined on the same sample space.

Distribution Function.

The distribution function of a random variable X, denoted by $F(x)$, is defined by $F(x) = P(X \leq x)$.

The function $F(x)$ gives the probability of

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the event that X takes a value less than or equal to a specified value x.

The distribution function is abbreviated as d.f.

A d.f. $F(x)$ has the following properties.

(i) $F(-\infty) = 0$, $F(+\infty) = 1$

(ii) $F(x)$ is a non-decreasing function of x.

i.e. $F(x_1) \leq F(x_2)$ if $x_1 \leq x_2$.

(iii) $F(x)$ is continuous at least on the right of each x.

Remark.

All random variables have distribution function. Distribution functions for different r.v.'s are distinguished by using the notation F_X, F_Y etc.

Discrete r.v.

A random variable X is said to be discrete if it can assume values which are finite or countably infinite.

The number of heads obtained in coin tossing experiments, the number of fatal accidents, the number of bacteria in 1cc of water, etc. are the examples of discrete r.v.

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Probability Function.

Let X be a discrete random variable taking on distinct values $x_1, x_2, \cdots, x_n \cdots$ Then the function, denoted by $p(x)$ or $f(x)$, and defined by

$$f(x_i) = \begin{cases} P(X = x_i) & \text{for } i = 1, 2, \cdots n \\ 0 & \text{for } x \neq x_i \end{cases}$$

is called the probability function of the r.v. X, and the values $x_1, x_2, \cdots, x_n, \cdots$ among which the total probability 1 is distributed, are called the probability points or jump points, $P(X = x_i)$ means the probability that the discrete r.v. X takes the value x_i.

The distribution function for a discrete r.v. is

$$F(x) = \sum_i f(x_i)$$

where sum is taken over all x_i that are less than or equal to x.

A probability distribution must satisfy the following two basic properties.

(i) $f(x_i) \geq 0, \quad \forall i$

(ii) $\sum_i f(x_i) = 1$.

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Probability Distribution.

Let X be a random variable assuming values $x_1, x_2, \cdots, x_n$ with probabilities $f(x_1), f(x_2), \cdots, f(x_n)$ then the ordered pairs $(x_i, f(x_i)) ; i = 1, 2, \cdots n$ is called the probability distribution of X.

Q 1.

Find the probability distribution and distribution function for the number of heads when 3 balanced coins are tossed.

Sol.

The sample space for this experiment is

$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

Let X be the r.v. that denotes the number of heads. Then the values of x are 0, 1, 2, and 3, and their probabilities are

$f(0) = P(X=0) = P(\{TTT\}) = 1/8$

$f(1) = P(X=1) = P(\{TTH\}, \{THT\}, \{HTT\}) = 3/8$

$f(2) = P(X=2) = P(\{HHT\}, \{HTH\}, \{THH\}) = 3/8$

$f(3) = P(X=3) = P(\{HHH\}) = 1/8$

Putting this information in the tabular form we obtain the desired probability distribution of X as

x_i	0	1	2	3
$f(x_i)$	$1/8$	$3/8$	$3/8$	$1/8$

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This result may be expressed by the equation as

$$f(x) = \frac{1}{8}\binom{3}{x} \quad \text{for } x = 0, 1, 2, 3.$$

For distribution function, we have

if $x < 0$ then $P(X < x) = 0$

if $0 \le x < 1$ then $P(X < x) = P(X = 0) = 1/8$

if $1 \le x < 2$ then $P(X < x) = P(X = 0) + P(X = 1)$

$$= 1/8 + 3/8 = 4/8$$

if $2 \le x < 3$ then

$$P(X < x) = P(X = 0) + P(X = 1) + P(X = 2)$$

$$= \frac{1}{8} + 3/8 + 3/8 = 7/8$$

if $x \geq 3$ then

$$P(X < x) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$

$$= \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = \frac{8}{8} = 1.$$

Hence, the desired distribution function is

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{8} & \text{for } 0 \le x < 1 \\ \frac{4}{8} & \text{for } 1 \le x < 2 \\ \frac{7}{8} & \text{for } 2 \le x < 3 \\ 1 & \text{for } x \geq 3 \end{cases}$$

Q2.

(b) Use the probability distribution to find the probabilities of obtaining

(i) a sum of 8 or 11

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(ii) a sum that is greater than 8

(iii) a sum that is greater than 5 but less than or equal to 10.

(a) Find the probability distribution of the sum of the dots when two fair dice are thrown.

Sol.

(a)

The sample space for this experiment is

$S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)$
$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)$
$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)$
$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)$
$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)$
$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$.

Let X be the r.v. that denotes the sum of dots which appear on the dice. Then the values of x are $2, 3, 4, \cdots, 12$, and their probabilities are

$f(2) = P((1,1)) = 1/36$

$f(3) = P((1,2), (2,1)) = 2/36$

$f(4) = P((1,3), (3,1), (2,2)) = 3/36$

$f(5) = P((1,4), (2,3), (3,2), (4,1)) = 4/36$

$f(6) = P((1,5), (2,4), (3,3), (4,2), (5,1)) = 5/36$

$f(7) = P((6,1), (2,5), (3,4), (4,3), (5,2), (6,1)) = 6/36$

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$f(8) = P((6,2), (2,6), (3,5), (4,4), (5,3)) = 5/36$

$f(9) = P((3,6), (4,5), (5,4), (6,3)) = 4/36$

$f(10) = P((4,6), (5,5), (6,4)) = 3/36$

$f(11) = P((5,6), (6,5)) = 2/36$

$f(12) = P((6,6)) = 1/36$.

Putting this information in tabular form, we obtain the desired probability distri. of X as

x_i	2	3	4	5	6	7	8	9	10	11	12
$f(x_i)$	$1/36$	$2/36$	$3/36$	$4/36$	$5/36$	$6/36$	$5/36$	$4/36$	$3/36$	$2/36$	$1/36$

This result may be expressed by the equation as

$$f(x) = \frac{6 - |7 - x|}{36}, \text{ for } x = 2, 3, 4, \ldots, 12.$$

(b)

(i) Sum of 8 or 11

$$P(X = 8 \text{ or } X = 11) = P(X=8) + P(X=11)$$
$$= f(8) + f(11)$$
$$= 5/36 + 2/36 = 7/36.$$

(ii) Sum that is greater than 8

$$P(X > 8) = P(X=9) + P(X=10) + P(X=11) + P(X=12)$$
$$= \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{10}{36}$$

(iii) Sum greater than 5 but less than or equal to 10.

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$$P(5 < X \leq 10) = P(X=6) + P(X=7) + P(X=8) + P(X=9) + P(X=10)$$

$$= \frac{5}{36} + \frac{6}{36} + \frac{5}{36} + \frac{4}{36} + \frac{3}{36}$$

$$P(5 < X \leq 10) = \frac{23}{36}$$

Q3.

A bag contains 4 red and 6 black balls. A sample of 4 balls is selected from the bag without replacement. Let X be the number of red balls. Find the p.d. for X.

Sol.

Given that

Number of red balls = 4

" " black " = 6

Total balls in bag = 10

Let X be a random variable that denotes the number of red balls. Then values of x are 0, 1, 2, 3, 4 and their probabilities are

$$f(0) = P(X=0) = \frac{\binom{4}{0}\binom{6}{4}}{\binom{10}{4}} = \frac{1 \times 15}{210} = \frac{15}{210}$$

$$f(1) = P(X=1) = \frac{\binom{4}{1}\binom{6}{3}}{\binom{10}{4}} = \frac{4 \times 20}{210} = \frac{80}{210}$$

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$$f(2) = P(X=2) = \frac{\binom{4}{2}\binom{6}{2}}{\binom{10}{4}} = \frac{(6)(15)}{210} = \frac{60}{210}$$

$$f(3) = P(X=3) = \frac{\binom{4}{3}\binom{6}{1}}{\binom{10}{4}} = \frac{(4)(6)}{210} = \frac{24}{210}$$

$$f(4) = P(X=4) = \frac{\binom{4}{4}\binom{6}{0}}{\binom{10}{4}} = \frac{(1)(1)}{210} = \frac{1}{210}$$

Putting this information in tabular form we obtain the desired p.d of X as.

x_i	0	1	2	3	4
$f(x_i)$	$15/210$	$8/210$	$60/210$	$24/210$	$1/210$

Q4.

A large store places its last 15 clock radios in a clearance scale. Unknown to any one, 5 of the radios are defective. If a customer tests 3 different clock radios selected at random, What is the p.d. of X = number of defective radios in the sample?

Sol.

Given that

Total number of clock radios = 15

number of defective clock radios = 5

" " tested " " = 3

Let X denote the number of defecti

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radios. Then the values of x are 0, 1, 2, 3 and their probabilities are

$$f(0) = P(X=0) = \frac{\binom{5}{0}\binom{10}{3}}{\binom{15}{3}} = \frac{(1)(120)}{455} = \frac{120}{455}$$

$$f(1) = P(X=1) = \frac{\binom{5}{1}\binom{10}{2}}{\binom{15}{3}} = \frac{(5)(45)}{455} = \frac{225}{455}$$

$$f(2) = \frac{\binom{5}{2}\binom{10}{1}}{\binom{15}{3}} = \frac{(10)(10)}{455} = \frac{100}{455}$$

$$f(3) = \frac{\binom{5}{3}\binom{10}{0}}{\binom{15}{3}} = \frac{(10)(1)}{455} = \frac{10}{455}$$

Putting this information in tabular form we obtain the desired p.d. of X as

x_i	0	1	2	3
$f(x_i)$	$\frac{120}{455}$	$\frac{225}{455}$	$\frac{100}{455}$	$\frac{10}{455}$

Q5.

Two socks are selected at random and removed in succession from a drawer containing five brown socks and three green socks. Find the distribution function for p.d. of W = number of brown socks.

Sol.

The sample space for this experiment is

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$S = \{GG, GB, BG, BB\}$.

Let W be a r.v that denotes the number of brown socks. Then the values of W are 0, 1, 2, and their probabilities are

$$f(0) = P(W=0) = \frac{\binom{5}{0}\binom{3}{2}}{\binom{8}{2}} = \frac{(1)(3)}{28} = \frac{3}{28}$$

$$f(1) = \frac{\binom{5}{1}\binom{3}{1}}{\binom{8}{2}} = \frac{(5)(3)}{28} = \frac{15}{28}$$

$$f(2) = \frac{\binom{5}{2}\binom{3}{0}}{\binom{8}{2}} = \frac{(10)(1)}{28} = \frac{10}{28}$$

Putting this information in tabular form we obtain the desired p.d. of W as

w_i	0	1	2
$f(w_i)$	$3/28$	$15/28$	$10/28$

Now we find distribution function as follow:

For $w < 0$, $P(W < w) = 0$

For $0 \le w < 1$, $P(W=0) = 3/28$

For $1 \le w < 2$, $P(W=0) + P(W=1) = \frac{3}{28} + \frac{15}{28}$

$$= \frac{18}{28} = \frac{9}{14}$$

For $2 \le w$, $P(W < w) = P(W=0) + P(W=1) + P(W=2)$

$$= \frac{3}{28} + \frac{15}{28} + \frac{10}{28} = \frac{28}{28} = 1.$$

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Thus, the desired distribution function is

$$F(x) = \begin{cases} 0 & \text{for } 0 > w \\ 3/28 & \text{for } 0 \le w < 1 \\ 15/28 & \text{for } 1 \le w < 2 \\ 28/28 = 1 & \text{for } w \geq 2 \end{cases}$$

Q6.

Find the distribution function of the total number of heads obtaining in four tosses of balanced coin?

Sol.

For this experiment number of elements in sample space are $2^4 = 16$.

Let X be r.v. that denotes the number of heads. Then the values of x are 0, 1, 2, 3, 4 and their probabilities are

$f(0) = P(X=0) = \frac{1}{16}$

$f(1) = P(X=1) = \frac{4}{16}$

$f(2) = P(X=2) = \frac{6}{16}$

$f(3) = P(X=3) = \frac{4}{16}$

$f(4) = P(X=4) = \frac{1}{16}$

$\because S = \{HHHH, HTTT, THTT, TTH$
$TTTH, HHTT, HT$
$HTTH, THTH, TT$
$THHT, HHHT,$
$HTHH, THHH,$
$HHTH, TTTT\}$

Putting this information in tabular form

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we obtain the desired p.d of X as

x_i	0	1	2	3	4
$f(x_i)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

Now we find the dis. function as follows.

For $x < 0$, $P(X < x) = 0$

For $0 \le x < 1$, $P(X < x) = P(X = 0) = \frac{1}{16}$

For $1 \le x < 2$; $P(X < x) = P(X = 0) + P(X = 1)$

$$= \frac{1}{16} + \frac{4}{16} = \frac{5}{16}$$

For $2 \le x < 3$; $P(X < x) = P(X = 0) + P(X = 1) + P(X = 2)$

$$= \frac{5}{16} + \frac{6}{16} = \frac{11}{16}$$

For $3 \le x < 4$; $P(X < x) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$

$$= \frac{11}{16} + \frac{4}{16} = \frac{15}{16}$$

For $4 \le x$; $P(X < x) = \sum_{i=1}^{4} P(X = x_i)$

$$= \frac{15}{16} + \frac{1}{16} = \frac{16}{16} = 1$$

Thus, the desired distribution function is

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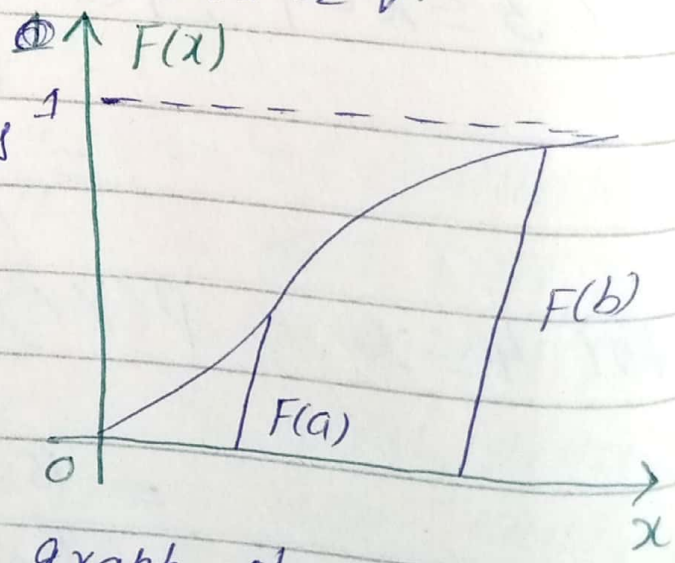
$$F(x) = \begin{cases} 0 & \text{for } x<0 \\ \frac{1}{16} & \text{for } 0 \le x < 1 \\ \frac{5}{16} & \text{for } 1 \le x < 2 \\ \frac{11}{16} & \text{for } 2 \le x < 3 \\ \frac{15}{16} & \text{for } 3 \le x < 4 \\ 1 & \text{for } 4 \le x \end{cases}$$

Continuous r.v.

A random variable X is said to be continuous if it can assume every possible value in an interval $[a, b]$, $a<b$, where a and b may be $-\infty$ and $+\infty$ respectively. The height of a person, the temperature at a place, the amount of rainfall, etc. are examples of continuous r.v.

A r.v. X may also be defined as continuous if its d.f. $F(x)$ is continuous and diff. everywhere except at isolated points in the given range. The graph of $F(x)$ has no jumps but is a continuous function for all x.



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Let the derivative of $F(x)$ be denoted by $f(x)$, i.e.

$$\frac{d}{dx} F(x) = f(x).$$

Since $F(x)$ is a non decreasing function of x, we have

(i) $f(x) \geq 0 \quad \forall x$

(ii) $F(x) = \int_{-\infty}^{x} f(x)\, dx, \quad \forall x$

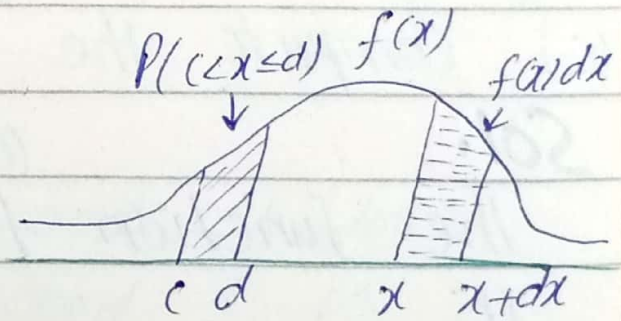
The function $f(x)$ is called the probability density function, abbreviated as p.d.f. of the r.v. X.

A p.d.f has the following properties:

(i) $f(x) \geq 0$ for all x.

(ii) $\int_{-\infty}^{\infty} f(x)\, dx = 1$

(iii) The probability that X takes on a value in the interval $[c, d]$, c, d is given by



$$P(c < x \leq d) = F(d) - F(c)$$

$$= \int_{-\infty}^{d} f(x)\, dx - \int_{-\infty}^{c} f(x)\, dx$$

$$= \int_{-\infty}^{d} f(x)\, dx + \int_{c}^{-\infty} f(x)\, dx$$

$$= \int_{c}^{-\infty} f(x)\, dx + \int_{-\infty}^{d} f(x)\, dx = \int_{c}^{d} f(x)\, dx$$

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which is the area under the curve $y=f(x)$ between $X=c$ and $X=d$.

Note.

Since for a continuous r.v. X, $P(X=x)=0$, for every x, the following four probab. are same.

$P(c \le X \le d)$, $P(c < X \le d)$, $P(c \le X < d)$, $P(c < X < d)$

They may be different with a discrete r.v.

Q1.

(a) Find the value of K so that the function $f(x)$ defined as follows, may be a density function

$$f(x) = \begin{cases} Kx & ; 0 \le x \le 2 \\ 0 & , \text{elsewhere} \end{cases}$$

(b) Find also the probability that both of the two sample values will exceed 1.

(c) Compute the dis. function $F(x)$.

Sol. (a)

The function $f(x)$ will be density fn. if

(i) $f(x) \geq 0$ for all x

This condition is satisfied when $K \geq 0$.

(ii) $\int_{-\infty}^{+\infty} f(x)\,dx = 1$

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$$\Rightarrow \int_{-\infty}^{\infty} Kx\,dx = 1 \quad \Rightarrow K\int_{-\infty}^{\infty} x\,dx = 1$$

$$\Rightarrow K\int_{-\infty}^{0} x\,dx + K\int_{0}^{2} x\,dx + K\int_{2}^{\infty} x\,dx = 1$$

$$\Rightarrow 0 + K\int_{0}^{2} x\,dx + 0 = 1$$

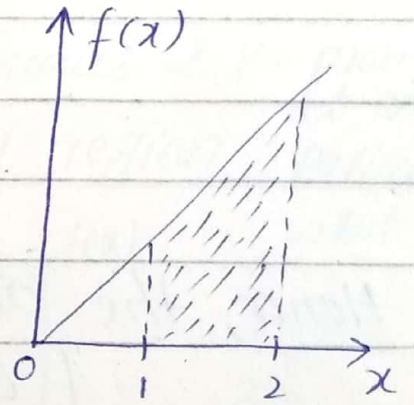
$$\Rightarrow K\left.\frac{x^2}{2}\right|_{0}^{2} = 1$$

$$\Rightarrow K\left(\frac{4}{2} - 0\right) = 1 \quad \Rightarrow 2K = 1 \quad \Rightarrow \boxed{K = 1/2}$$

Hence, $f(x) = \begin{cases} \frac{1}{2}x, & 0 \le x \le 2 \\ 0, & \text{eleswhere} \end{cases}$

(b)

$P(X > 1)$ = area of shaded region



$$= \int_{1}^{2} f(x)\,dx$$

$$= \frac{1}{2}\int_{1}^{2} x\,dx = \frac{1}{2}\left.\frac{x^2}{2}\right|_{1}^{2}$$

$$= \frac{1}{4}(4 - 1) = 3/4$$

Hence, the probab. that both of two sample values will exceed 1 is

$$\frac{3}{4} \times \frac{3}{4} = \frac{9}{16}.$$

(C)

For distribution function,

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$$F(x) = \int_{-\infty}^{x} f(x)\,dx.$$

For $-\infty < x < 0$, $F(x) = \int_{-\infty}^{x} 0\,dx = 0$

For $0 \le x \le 2$, $F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{x} \frac{1}{2} x\,dx$$

$$= \frac{1}{2}\left|\frac{x^2}{2}\right|_0^x = \frac{x^2}{4}$$

For $x > 2$, $F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx + \int_{2}^{x} f(x)\,dx$

$$= 0 + \frac{1}{2}\int_0^2 x\,dx + 0$$

$$= \frac{1}{2}\left|\frac{x^2}{2}\right|_0^2 = \frac{1}{4}(4-0) = 1$$

Hence, the desired distribution function is

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{x^2}{4} & \text{for } 0 \le x \le 2 \\ 1 & \text{for } x > 2 \end{cases}$$

Q2.

A r.v. X is of continuous type with p.d.f

$$f(x) = \begin{cases} 2x & , 0 < x < 1 \\ 0 & , \text{elsewhere} \end{cases}$$

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Find (i) $P(X=\frac{1}{2})$ (ii) $P(X\le\frac{1}{2})$ (iii) $P(X>\frac{1}{4})$ (iv) $P(\frac{1}{4}\le X<\frac{1}{2})$, and (v) $P(X\le\frac{1}{2}\,/\,\frac{1}{3}\le X\le\frac{2}{3})$

Sol.

First we check p.d.f.

(i) $f(x)\ge 0$ (clearly)

(ii) $$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx$$

$$= 0 + \int_{0}^{1} 2x\,dx + 0 = x^2\Big|_0^1 = 1-0$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$\Rightarrow$ given $f(x)$ is p.d.f.

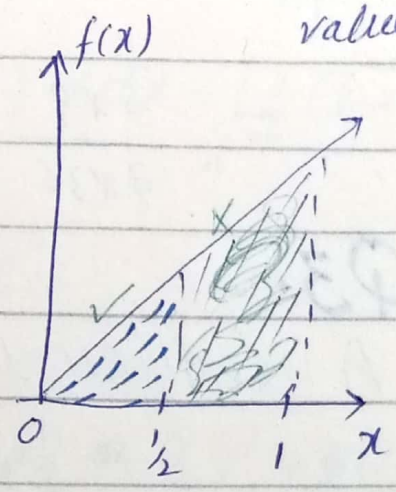
(i) $P(X=\frac{1}{2}) = 0$

because for continuous r.v. prob. at any particular value.

(ii) $P(X\le\frac{1}{2})$ = area of shaded region.

$$= \int_0^{1/2} 2x\,dx$$

$$= x^2\Big|_0^{1/2} = \frac{1}{4}$$

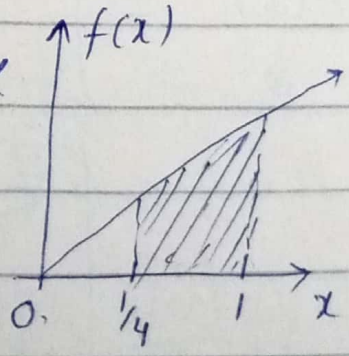


(iii) $P(X>\frac{1}{4})$

= area of shaded region.

$$= \int_{-\infty}^{1/4} f(x)\,dx + \int_{1/4}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx$$

$$= \int_{1/4}^{1} 2x\,dx = x^2\Big|_{1/4}^{1} = 1-\frac{1}{16} = \frac{15}{16}$$



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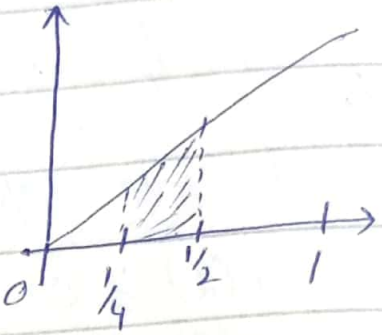
(iv) $P\left(\frac{1}{4} \le X < \frac{1}{2}\right)$ = area of shaded region.



$$= \int_{1/4}^{1/2} 2x\,dx$$

$$= x^2 \Big|_{\frac{1}{4}}^{\frac{1}{2}}$$

$$= \frac{1}{4} - \frac{1}{16} = \frac{4-1}{16} = \frac{3}{16}$$

(v) By conditional probability

$$P\left(X \le \frac{1}{2} \Big/ \frac{1}{3} \le X \le \frac{2}{3}\right) = \frac{P\left(\frac{1}{3} \le X \le \frac{1}{2}\right)}{P\left(\frac{1}{3} \le X \le \frac{2}{3}\right)}$$

$$= \frac{\int_{1/3}^{1/2} 2x\,dx}{\int_{1/3}^{2/3} 2x\,dx}$$

$$= \frac{x^2 \Big|_{1/3}^{1/2}}{x^2 \Big|_{1/3}^{2/3}} = \frac{\frac{1}{4} - \frac{1}{9}}{\frac{4}{9} - \frac{1}{9}} = \frac{\frac{9-4}{36}}{\frac{4-1}{9}}$$

$$= \frac{5 \times 9}{3 \times 36} = \frac{5}{3 \times 4} = \frac{5}{12}.$$

Q3.

A continuous rand. v. X has the d.f.

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{2x^2}{5} & \text{for } 0 < x \le 1 \\ -\frac{3}{5} + \frac{2}{5}\left(3x - \frac{x^2}{2}\right) & \text{for } 1 < x < 2 \\ 1 & \text{for } x > 2 \end{cases}$$

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Find the p.d. and $P(|X| < 1.5)$

Sol. By definition of p.d.

$$f(x) = \frac{d}{dx} F(x)$$

$$\Rightarrow f(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{4}{5}x & \text{for } 0 < x \le 1 \\ \frac{2}{5}(3-x) & \text{for } 1 < x \le 2 \\ 0 & \text{for } x > 0 \end{cases}$$

Now, we find $P(|X| < 1.5)$

$$P(|X| < 1.5) = P(-1.5 < X < 1.5)$$

$$= \int_{-\infty}^{-1.5} f(x)\,dx + \int_{-1.5}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{1.5} f(x)\,dx$$

$$= \int_{-\infty}^{-1.5} 0\,dx + \int_{-1.5}^{0} 0\,dx + \int_{0}^{1} \frac{4}{5}x\,dx + \int_{1}^{1.5} \frac{2}{5}(3-x)\,dx$$

$$= \frac{4}{5}\left|\frac{x^2}{2}\right|_0^1 + \frac{2}{5}\left|3x - \frac{x^2}{2}\right|_1^{1.5}$$

$$= \frac{4}{5}\left(\frac{1}{2}\right) + \frac{2}{5}\left(3(1.5) - 3(1) - \frac{(1.5)^2}{2} + \frac{(1)^2}{2}\right)$$

$$= \frac{2}{5} + \frac{2}{5}\left(\frac{7}{8}\right) = \frac{2}{5} + \frac{7}{20} = 3/4$$

$$= 0.75$$

req. Ans.

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Q4. A continuous random variable has the p.d.f. as follows.

$$f(x) = \begin{cases} x/2 & \text{for } 0 < x \le 1 \\ \frac{1}{4}(3-x) & \text{for } 1 < x \le 2 \\ \frac{1}{4} & \text{for } 2 < x \le 3 \\ \frac{1}{4}(4-x) & \text{for } 3 \le x \le 4 \\ 0 & \text{elsewhere} \end{cases}$$

Compute $P(X \geq 3)$, $P(X=2)$, $P(|X| < 1.5)$ and $P(1 < X < 3)$.

Sol.

$$P(X \geq 3) = \int_3^4 \frac{1}{4}(4-x)\,dx$$

$$= \frac{1}{4}\left[4x - \frac{x^2}{2}\right]_3^4 = \frac{1}{4}\left[\cancel{16} - \cancel{12} - \cancel{8} + \frac{9}{2}\right]$$

$$= \frac{1}{4}\left(\frac{-4+9}{2}\right) = \frac{1}{4}\left(\frac{-8+9}{2}\right) = \frac{1}{8}$$

$P(X=2) = 0$, because for a continuous random variable the probability at any particular value is equal to zero.

$$P(|X| < 1.5) = P(-1.5 < X < 1.5)$$

$$= \int_{-\infty}^{-1.5} f(x)\,dx + \int_{-1.5}^{0} f(x)\,dx + \int_0^1 f(x)\,dx + \int_1^{1.5} f(x)\,dx$$

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$$= \int_{-\infty}^{-1.5} 0\,dx + \int_{-1.5}^{0} 0\,dx + \int_{0}^{1} \frac{x}{2}\,dx + \int_{1}^{1.5} \frac{1}{4}(3-x)\,dx$$

$$= \frac{x^2}{4}\Big|_0^1 + \frac{1}{4}\left(3x - \frac{x^2}{2}\right)\Big|_1^{1.5}$$

$$= \frac{1}{4} + \frac{1}{4}\left[3(1.5) - 3 - \frac{(1.5)^2}{2} + \frac{1}{2}\right]$$

$$= \frac{1}{4} + \frac{1}{4}\left(\frac{7}{8}\right) = \frac{15}{32} = 0.468750000$$

$$P(1 < X < 3) = \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx$$

$$= \int_1^2 \frac{(3-x)}{4}\,dx + \int_2^3 \frac{1}{4}\,dx$$

$$= \frac{1}{4}\left|3x - \frac{x^2}{2}\right|_1^2 + \frac{1}{4}x\Big|_2^3$$

$$= \frac{1}{4}\left(6 - 3 - 2 + \frac{1}{2}\right) + \frac{1}{4}(3-2)$$

$$= \frac{1}{4}\left(\frac{3}{2} + 1\right) = \frac{1}{4}\left(\frac{5}{2}\right) = \frac{5}{8}$$

Q5. A continuous r.v. has the p.d.f.

$$f(x) = \begin{cases} A(2-x)(2+x) & ; 0 \le x \le 2 \\ 0 & , \text{elsewhere} \end{cases}$$

Find (i) the value of A (ii) $P(X = \frac{1}{2})$ (iii) $P(X \le 1)$ (iv) $P(X \ge 2)$ (v) $P(1 \le x \le 2)$

Sol.

(i) For p.d.f, we have,

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$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx + \int_{2}^{+\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} 0\,dx + \int_{0}^{2} A(4-x^2)\,dx + \int_{2}^{\infty} 0\,dx = 1$$

$$\Rightarrow A\left[4x - \frac{x^3}{3}\right]_0^2 = 1$$

$$\Rightarrow A\left[8 - 0 - \frac{8}{3} + 0\right] = 1$$

$$\Rightarrow A\left(\frac{24-8}{3}\right) = 1 \Rightarrow A\left(\frac{16}{3}\right) = 1$$

$$\Rightarrow \boxed{A = 3/16}$$

Hence $f(x) = \begin{cases} \frac{3}{16}(4-x^2) & ; 0 \le x \le 2. \\ 0 & ; \text{elsewhere} \end{cases}$

(ii) $P(X = \frac{1}{2}) = 0$; because for a continuous r.v. the prob. at any particular value is equal to zero.

(iii) $P(X \le 1) = \int_0^1 f(x)\,dx$

$$= \frac{3}{16}\int_0^1 (4-x^2)\,dx$$

$$= \frac{3}{16}\left(4x - \frac{x^3}{3}\right)\Big|_0^1$$

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$$= \frac{3}{16}\left(4 - \frac{1}{3}\right) = \frac{3}{16}\left(\frac{12-1}{3}\right) = \frac{11}{16}$$

(iv) $P(X \geq 2) = 0$

as outside the range 0 to 2 $f(x) = 0$.

(v) $P(1 \leq X \leq 2) = \frac{3}{16}\int_1^2 (4 - x^2)\,dx$

$$= \frac{3}{16}\left(4x - \frac{x^3}{3}\right)\Big|_1^2$$

$$= \frac{3}{16}\left(8 - 4 - \frac{8}{3} + \frac{1}{3}\right)$$

$$= \frac{3}{16}\left(4 - \frac{8}{3} + \frac{1}{3}\right) = \frac{3}{16}\left(\frac{12-8+1}{3}\right)$$

$$= \frac{5}{16}$$

Q6. Let X be a continous r.v. with p.d.f.

$$f(x) = \begin{cases} 6x(1-x) & ; 0 \leq x \leq 1 \\ 0 & , \text{elsewhere} \end{cases}$$

(a) Determine the distribution function of X.

(b) Find a number b such that $P(X < b) = 2P(X > b)$.

Sol.

(a) Distribution function

$$F(x) = \int_{-\infty}^{x} f(x)\,dx. \quad \forall x.$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx.$$

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$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{x} 6x(1-x)\,dx$$

$$= 6\int_{0}^{x} (x - x^2)\,dx = 6\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_{0}^{x}$$

$$= 6\left(\frac{x^2}{2} - \frac{x^3}{3}\right) = 6\left(\frac{3x^2 - 2x^3}{6}\right)$$

$$\Rightarrow F(x) = 3x^2 - 2x^3.$$

$$P(X<b) = \int_{0}^{b} f(x)\,dx = 6\int_{0}^{b} (x - x^2)\,dx$$

$$= 6\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_{0}^{b}$$

$$= (3x^2 - 2x^3)\Big|_{0}^{b} = 3b^2 - 2b^3$$

$$\Rightarrow P(X<b) = 3b^2 - 2b^3 \quad \text{——— (1)}$$

and $P(X>b) = 1 - P(X<b)$

$$= 1 - 3b^2 + 2b^3 \quad \text{——— (2)}$$

Subsituting (1) and (2) in given Condition $P(X<b) = 2P(X>b)$, we get

$$3b^2 - 2b^3 = 2 - 6b^2 + 4b^3$$

$$\Rightarrow 6b^2 - 4b^3 + 3b^2 - 2b^3 = 2$$

$$\Rightarrow 9b^2 - 6b^3 = 2$$

$$\Rightarrow 6b^3 - 9b^2 + 2 = 0$$

$$\Rightarrow b^3 - \frac{3}{2}b^2 + \frac{1}{3} = 0$$

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we solve this equation by Newton's Raphson method as follows

$$f(b) = b^3 - \frac{3}{2}b^2 + \frac{1}{3} = 0$$

$$b_{n+1} = b_n - \frac{f(b_n)}{f'(b_n)}$$

For $n=0$

$$b_1 = b_0 - \frac{f(b_0)}{f'(b_0)} = b_0 - \frac{b_0^3 - \frac{3}{2}b_0^2 + \frac{1}{3}}{3b_0^2 - 3b_0}$$

Let us take $b_0 = 0.5$, then

$$b_1 = 0.5 - \frac{(0.5)^3 - \frac{3}{2}(0.5)^2 + \frac{1}{3}}{3(0.5)^2 - 3(0.5)}$$

$$= 0.5 + \frac{1}{9} = \frac{11}{18} = 0.611111111$$

Hence req. number $b = 0.6111$.

Q7. A/2011.

If X has the p.d.f

$$f(x) = \begin{cases} K\exp(-3x) & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

(i) Find K and $P(0.5 \le X \le 1)$

(ii) Find the d.f. of X and use it to reevaluate $P(0.5 \le X \le 1)$.

Sol. (i)

For p.d.f, we must have

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

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$$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} K e^{-3x}\,dx = 1$$

$$\Rightarrow K\int_{0}^{\infty} e^{-3x}\,dx = 1$$

$$\Rightarrow K\left.\frac{e^{-3x}}{-3}\right|_{0}^{\infty} = 1 \quad \Rightarrow K\left(0+\frac{1}{3}\right) = 1$$

$$\Rightarrow K = 3$$

Hence, $f(x) = \begin{cases} 3e^{-3x} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$

$$P(0.5 \le X \le 1) = \int_{0.5}^{1} 3e^{-3x}\,dx$$

$$= -e^{-3x}\Big|_{0.5}^{1}$$

$$= -e^{-3} + e^{-1.5}$$

$$= -0.049787068 + 0.223130160$$

$$= 0.173343092$$

(ii)

By definition of d.f. of r.v. X

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= \int_{0}^{x} 3e^{-3x}\,dx$$

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$$= -e^{-3x}\Big|_0^x = -e^{-3x} + 1$$

$$\Rightarrow F(x) = 1 - e^{-3x}$$

$$P(0.5 \le X \le 1) = F(1) - F(0.5)$$

$$= (1 - e^{-3}) - (1 - e^{-1.5})$$

$$= (1 - 0.49787068) - (1 - 0.22313016)$$

$$= 0.950212932 - 0.776869840$$

$$= 0.173343092.$$

Q8.

Find distribution function of Q4.

Sol.

By definition

$$F(x) = \int_{-\infty}^{x} f(x)\,dx \quad \forall x.$$

For $x < 0$, $F(x) = \int_{-\infty}^{0} f(x)\,dx = 0$

For $0 < x \le 1$, $F(x) = \int_0^x f(x)\,dx$

$$= \int_0^x \frac{x}{2}\,dx = \frac{x^2}{4}\Big|_0^x = \frac{x^2}{4}$$

For $1 < x \le 2$, $F(x) = \int_0^1 f(x)\,dx + \int_1^x f(x)\,dx$

$$= \int_0^1 \frac{x}{2}\,dx + \int_1^x \frac{1}{4}(3-x)\,dx$$

$$= \frac{x^2}{4}\Big|_0^1 + \frac{1}{4}\left(3x - \frac{x^2}{2}\right)\Big|_1^x$$

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$$= \frac{1}{4} + \frac{1}{4}\left(3x - \frac{x^2}{2}\right) - \frac{1}{4}\left(3 - \frac{1}{2}\right)$$

$$= \frac{1}{4} + \frac{1}{4}\left(3x - \frac{x^2}{2}\right) - \frac{1}{4}\left(\frac{5}{2}\right)$$

$$= \frac{1}{4}\left(1 - \frac{5}{2}\right) + \frac{1}{4}\left(3x - \frac{x^2}{2}\right)$$

$$= \frac{1}{4}\left(-3/2\right) + \frac{1}{4}\left(3x - x^2/2\right)$$

$$= \frac{1}{4}\left(3x - \frac{x^2}{2}\right) - \frac{3}{8}$$

For $2 < x \leq 3$; $F(x) = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^x f(x)\,dx$

$$= \int_0^1 \frac{x}{2}\,dx + \int_1^2 \frac{1}{4}(3-x)\,dx + \int_2^x \frac{1}{4}\,dx$$

$$= \frac{1}{4} + \frac{1}{4}\left(3x - \frac{x^2}{2}\right)\Big|_1^2 + \frac{x}{4}\Big|_2^x$$

$$= \frac{1}{4} + \frac{1}{4}\left(6 - 3 - 2 + \frac{1}{2}\right) + \frac{x}{4} - \frac{2}{4}$$

$$= \frac{1}{4} + \frac{1}{4}\left(\frac{3}{2}\right) + \frac{x}{4} - \frac{1}{2}$$

$$= \frac{1}{4} + \frac{3}{8} + \frac{x}{4} - \frac{1}{2}$$

$$= \frac{2 + 3 + 2x - 4}{8}$$

$$= \frac{2x + 1}{8} = \frac{1}{4}x + \frac{1}{8}$$

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$$= \frac{1}{4}\left(x + \frac{1}{2}\right)$$

For $3 < x \leq 4$:

$$F(x) = \int_0^1 \frac{x}{2}\,dx + \int_1^2 \frac{1}{4}(3-x)\,dx + \int_2^3 \frac{1}{4}\,dx + \int_3^x \frac{1}{4}(4-x)\,dx$$

$$= \frac{1}{4} + \frac{3}{8} + \frac{3}{4} - \frac{1}{2} + \frac{1}{4}\left(4x - \frac{x^2}{2}\right)\Big|_3^x$$

$$= \frac{1}{4} + \frac{3}{8} + \frac{3}{4} - \frac{1}{2} + \frac{1}{4}\left(4x - \frac{x^2}{2}\right) - \frac{1}{4}\left(12 - \frac{9}{2}\right)$$

$$= \frac{1}{8}\left[2 + 3 + 6 - 4 + 2\left(4x - \frac{x^2}{2}\right) - 2\left(\frac{24-9}{2}\right)\right]$$

$$= \frac{1}{8}\left[7 + 2\left(4x - \frac{x^2}{2}\right) - 15\right]$$

$$= \frac{1}{8}\left[-8 + 2\left(4x - \frac{x^2}{2}\right)\right]$$

$$= \frac{1}{4}\left(4x - \frac{x^2}{2}\right) - 1$$

For $x \geq 4$

$$F(x) = \int_0^1 \frac{x}{2}\,dx + \int_1^2 \frac{1}{4}(3-x)\,dx + \int_2^3 \frac{1}{4}\,dx$$

$$+ \int_3^4 \frac{1}{4}(4-x)\,dx + \int_4^x 0\,dx$$

$$= \frac{1}{4}\left(4x - \frac{x^2}{2}\right) - 1 \quad \text{for } x = 4$$

$$= \frac{1}{4}[16 - 8] - 1$$

$$= \frac{1}{4}(8) - 1 = 2 - 1 = 1$$

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Joint Distribution.

The distribution of two or more r.variables which are observed simultaneously when an experiment is performed, is called their joint distribution.

Bivariate Distribution Function.

Let X and Y be two r.v.'s defined on the same sample space S. Then the function $F(x, y)$ defined by

$$F(x, y) = P(X \leq x \text{ and } Y \leq y),$$

where $F(x, y)$ gives the probability that X will take on a value less than or equall to x and, at the same time, Y will take on a value less than or equall to y, is called a bivariate or joint distribution function of X and Y

Properties.

(i) $F(x, -\infty) = F(-\infty, y) = 0$

$F(x, \infty) = F(\infty, y) = 1$

(ii) $F(x, y)$ is a non decreasing function of x and y, and is continuous on right.

(iii) If $x_1 < x_2$ and $y_1 < y_2$, then

$$P(x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2) = P(X < x_2, Y < y_2) - P(X < x_2, Y < y_1) - P(X < x_1, Y < y_2) + P(X < x_1, Y < y_1)$$

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$= F(x_2, y_2) - F(x_2, y_1) - F(x_1, y_2) + F(x_1, y_1)$.

A bivariate distribution may be discrete when the possible values of (X, Y) are finite or countably infinite.

It is continuous if (X, Y) can assume all values in some non-countable set of the plane.

A bivariate d. is said to be mixed when one r.v. is discrete and the other is continuous.

Bivariate Probability Function.

Let X and Y be two discrete r.v's. defined on the same sample space S, X taking the values $x_1, x_2, \cdots x_m$ and Y taking the values $y_1, y_2, \cdots, y_n$. Then the probability that X takes on the values x_i and, at the same time, Y takes on the values y_j, denoted by $f(x_i, y_j)$, is defined as the joint probability function.

It is also called bivariate probability function and is given by

$$f(x_i, y_j) = P(X = x_i \text{ and } Y = y_j), \quad \begin{matrix} i = 1, 2, \cdots m \\ j = 1, 2, \cdots n \end{matrix}$$

The joint prob. dis. consisting of all pairs of values (x_i, y_j) and their associated probabilities $f(x_i, y_j)$.

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Joint Probability Dis. of X and Y

X \ Y	y_1	y_2	$\cdots$	y_j	$\cdots$	y_n	$P(X=x_i)$
x_1	$f(x_1,y_1)$	$f(x_1,y_2)$	$\cdots$	$f(x_1,y_j)$	$\cdots$	$f(x_1,y_n)$	$g(x_1)$
x_2	$f(x_2,y_1)$	$f(x_2,y_2)$	$\cdots$	$f(x_2,y_j)$	$\cdots$	$f(x_2,y_n)$	$g(x_2)$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
x_i	$f(x_i,y_1)$	$f(x_i,y_2)$	$\cdots$	$f(x_i,y_j)$	$\cdots$	$f(x_i,y_n)$	$g(x_i)$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
x_m	$f(x_m,y_1)$	$f(x_m,y_2)$	$\cdots$	$f(x_m,y_j)$	$\cdots$	$f(x_m,y_n)$	$g(x_m)$
$P(Y=y_j)$	$h(y_1)$	$h(y_2)$	$\cdots$	$h(y_j)$	$\cdots$	$h(y_n)$	1

A joint probability function has the following properties:

(i) $f(x_i, y_j) > 0 \quad \forall (x_i, y_j)$

(ii) $\sum_i \sum_j f(x_i, y_j) = 1$

Marginal Probability Function.

The ~~original~~ individual prob. functions of X and Y are called marginal prob. function.

The marginal p.f of X is

$$g(x_i) = \sum_{j=1}^{n} f(x_i, y_j)$$

$$= P(X = x_i)$$

and the marginal p.f of Y is

$$h(y_j) = \sum_{i=1}^{m} f(x_i, y_j) = P(Y = y_j)$$

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Conditional Prob. function.

Let X and Y be two discrete r.v.'s with joint prob. function $f(x,y)$. Then the conditional probability function for X given $Y=y$, denoted by $f(x|y)$, is defined by

$$f(x_i|y_j) = P(X=x_i | Y=y_j)$$
$$= \frac{P(X=x_i \text{ and } Y=y_j)}{P(Y=y_j)} \qquad h(y_j) \neq 0$$

$$\Rightarrow f(x_i|y_j) = \frac{f(x_i, y_j)}{h(y_j)} \quad \text{for } i=1,2,\ldots; j=1,2,\ldots$$

where $h(y)$ is the marginal probability.

Similarly, the conditional probability function for Y given $X=x$ is

$$f(y_j|x_i) = P(Y=y_j | X=x_i)$$

$$= \frac{P(Y=y_j \text{ and } X=x_i)}{P(X=x_i)}$$

$$\Rightarrow f(y_j|x_i) = \frac{f(x_i, y_j)}{g(x_i)} \qquad g(x_i) \neq 0.$$

where $g(x)$ is the marginal probability

Independence.

Two discrete r.v.'s X and Y are said to be independent if and only if, for all possible pairs of values (x_i, y_j) the joint p.f. $f(x,y)$ can be expressed as the product

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of the two marginal p.functions.

$$f(x,y) = g(x)\,h(y)$$

Q1.

An urn contains 3 black, 2 red and 3 green balls and 2 balls are selected at random from it. If X is the number of black balls and Y is the number of red balls selected, then find

(i) the joint p.f $f(x,y)$

(ii) $P(X+Y \le 1)$

(iii) $P(X=0 \mid Y=1)$

(iv) Are X and Y are independent?

Sol.

The sample space for this experiment contains $\binom{8}{2} = 28$ sample points.

The possible values of x are 0, 1, 2 and for y 0, 1, 2.

Now we find $f(x,y)$ as follows:

$$f(0,0) = \frac{\binom{3}{0}\binom{2}{0}\binom{3}{2}}{\binom{8}{2}} = \frac{(1)(1)(3)}{28} = \frac{3}{28}$$

$$f(0,1) = \frac{\binom{3}{0}\binom{2}{1}\binom{3}{1}}{\binom{8}{2}} = \frac{(1)(2)(3)}{28} = \frac{6}{28}$$

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$$f(0,2) = \frac{\binom{3}{0}\binom{2}{2}\binom{3}{0}}{\binom{8}{2}} = \frac{(1)(1)(1)}{28} = \frac{1}{28}$$

$$f(1,0) = \frac{\binom{3}{1}\binom{2}{0}\binom{3}{1}}{\binom{8}{2}} = \frac{(3)(1)(3)}{28} = \frac{9}{28}$$

$$f(1,1) = \frac{\binom{3}{1}\binom{2}{1}\binom{3}{0}}{\binom{8}{2}} = \frac{(3)(2)(1)}{28} = \frac{6}{28}$$

$$f(2,0) = \frac{\binom{3}{2}\binom{2}{0}\binom{3}{0}}{\binom{8}{2}} = \frac{(3)(1)(1)}{28} = \frac{3}{28}$$

Putting this information in tabular form, we get

X \ Y	0	1	2	g(x)
0	3/28	6/28	1/28	10/28
1	9/28	6/28	0	15/28
2	3/28	0	0	3/28
h(y)	15/28	12/28	1/28	1

This result may be expressed by the equation as

$$f(x,y) = \frac{\binom{3}{x}\binom{2}{y}\binom{3}{2-x-y}}{\binom{8}{2}} \quad ; x = 0,1,2 \quad y = 0,1,2$$

but $x+y \leq 2$

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(ii)

$$P(X+Y \le 1) = P(X=0, Y=0) + P(X=0, Y=1) + P(X=1, Y=0)$$

$$= f(0,0) + f(0,1) + f(1,0)$$

$$= \frac{3}{28} + \frac{6}{28} + \frac{9}{28} = \frac{18}{28} = \frac{9}{14}$$

(iii)

$$P(X=0 / Y=1) = f(0/1)$$

$$= \frac{f(0,1)}{h(1)} \quad \text{——— (1)}$$

where $h(1) = \sum_{x=0}^{2} f(x,1)$

$$= f(0,1) + f(1,1) + f(2,1)$$

$$= \frac{6}{28} + \frac{6}{28} + 0 = \frac{12}{28}$$

Thus (1) $\Rightarrow P(X=0 / Y=1) = \dfrac{6/28}{12/28} = \dfrac{6}{12} = 1/2$

In particular, (iv)

$$g(0) = \sum_{y=0}^{2} f(0, y) = f(0,0) + f(0,1) + f(0,2)$$

$$= \frac{3}{28} + \frac{6}{28} + \frac{1}{28} = \frac{10}{28}$$

and

$h(1) = 12/28$

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$\Rightarrow f(0,1) \neq g(0)\, h(1)$

$\Rightarrow$ X and Y are not independent.

Q2.

The joint p.d. of two discrete r.v.'s X and Y is given by

$$f(x,y) = \frac{xy^2}{30} \quad \text{for } x = 1,2,3 \text{ and } y = 1,2$$

Are X and Y are independent?

Sol.

The marginal p.d. of X is

$$g(x) = \sum_{y=1}^{2} f(x,y)$$

$$= \sum_{y=1}^{2} \frac{xy^2}{30} = \frac{x}{30} + \frac{4x}{30} = \frac{5x}{30} = \frac{x}{6}$$

and marginal p.d. of Y is

$$h(y) = \sum_{x=1}^{3} f(x,y)$$

$$= \sum_{x=1}^{3} \frac{xy^2}{30} = \frac{y^2}{30} + \frac{2y^2}{30} + \frac{3y^2}{30} = \frac{6}{30} y^2$$

$$\Rightarrow h(y) = y^2/5$$

$$\Rightarrow f(x,y) = \frac{x}{6} \cdot y^2/5 = \frac{xy^2}{30}$$

Hence X and Y are independent.

Next, onward from page 243 book S.Ch.

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A/2013

V.V.V.V.Imp

Chebyshev's Inequality.

If X is a r.v. having mean μ and variance $\sigma^2 > 0$, and K is any +ve const, then the probability that a value of X falls within K standard deviations of the mean is at least $(1 - 1/K^2)$.

That is

$$P(\mu - K\sigma < X < \mu + K\sigma) \geq 1 - 1/K^2$$

or $P(|X - \mu| \geq K\sigma) \leq 1/K^2$.

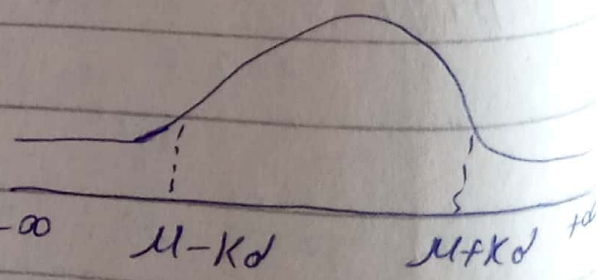
Proof

By definition, we have

$$\sigma^2 = E(X-\mu)^2$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

Dividing the range into three disjoint parts $(-\infty, \mu - K\sigma)$, $(\mu - K\sigma, \mu + K\sigma)$ and $(\mu + K\sigma, +\infty)$, we get



$$\sigma^2 = \int_{-\infty}^{\mu - K\sigma} (x-\mu)^2 f(x)\,dx + \int_{\mu - K\sigma}^{\mu + K\sigma} (x-\mu)^2 f(x)\,dx + \int_{\mu + K\sigma}^{+\infty} (x-\mu)^2 f(x)\,dx$$

Dropping the middle term, we get

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$$\sigma^2 \geq \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)\,dx + \int_{\mu+k\sigma}^{+\infty} (x-\mu)^2 f(x)\,dx$$

Since $|x-\mu| \geq k\sigma$, therefore, we have

$(x-\mu)^2 \geq k^2\sigma^2$,

whenever $x-\mu \geq k\sigma \Rightarrow x \geq \mu+k\sigma$

or $x-\mu \leq -k\sigma \Rightarrow x \leq \mu-k\sigma$.

So

$$\sigma^2 \geq \int_{-\infty}^{\mu-k\sigma} k^2\sigma^2 f(x)\,dx + \int_{\mu+k\sigma}^{+\infty} k^2\sigma^2 f(x)\,dx$$

$$\Rightarrow k^2\sigma^2\left(\int_{-\infty}^{\mu-k\sigma} f(x)\,dx + \int_{\mu+k\sigma}^{+\infty} f(x)\,dx\right) \leq \sigma^2$$

$$\Rightarrow \int_{-\infty}^{\mu-k\sigma} f(x)\,dx + \int_{\mu+k\sigma}^{+\infty} f(x)\,dx \leq 1/k^2$$

Since the sum of two integrals on the LHS is the probability that X will take a value less than or equal to $\mu-k\sigma$. or greater than equal to $\mu+k\sigma$.

Therefore,

$$P(\mu-k\sigma < X < \mu+k\sigma) = \int_{\mu-k\sigma}^{\mu+k\sigma} f(x)\,dx \leq 1/k^2$$

$$\Rightarrow P(\mu-k\sigma < X < \mu+k\sigma) \geq 1-1/k^2$$

or $P(|x-\mu| \geq k\sigma) \leq 1/k^2$

as required

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: DISTRIBUTION FUNCTION :

If X is a random variable and the values of its probability density at t is $f(t)$ then the function is given by

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt \quad \text{for } -\infty < x < +\infty$$

is called distribution function.

REMARK:

i)- $P(a \le X \le b) = F(b) - F(a)$

ii)- $f(x) = \frac{d}{dx} F(x)$

* ADE

A/2011 **EXAMPLE # 3.10 (JOHN E FRUENDS)**

Find the distribution function of random variable X

$$f(x) = \begin{cases} 3e^{-3x} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

and use it to re-evaluate $P(0.5 \le x \le 1)$

Sol:

For $x > 0$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$= \int_{-\infty}^{0} f(t)\,dt + \int_{0}^{x} f(t)\,dt$$

$$= \int_{-\infty}^{0} 3e^{-3t}\,dt + \int_{0}^{x} 3e^{-3t}\,dt$$

$$= 0 + 3\left|\frac{e^{-3t}}{-3}\right|_{0}^{x}$$

$$= \frac{-3}{3}\left[e^{-3x} - e^{0}\right]$$

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$$= -\left[e^{-3x} - 1\right]$$

$$= 1 - e^{-3x}$$

Since, Distribution function can be written as

$$F(x) = \begin{cases} 1 - e^{-3x} & ; \quad x > 0 \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

Now, we evaluate

$$P(0.5 \le x \le 1) = F(1) - F(0.5)$$

$$= \left(1 - e^{-3(1)}\right) - \left(1 - e^{-3(0.5)}\right)$$

$$= 1 - e^{-3} - 1 + e^{-1.5}$$

$$= 0.173 \text{ Ans}$$

EXAMPLE # 3.11 (JOHN E FRUENDS)

Find a probability density function for the random variable whose distribution function is given by

$$F(x) = \begin{cases} 0 & \text{for} \quad x \le 0 \\ x & \text{for} \quad 0 < x < 1 \\ 1 & \text{for} \quad x \geq 1 \end{cases}$$

Sol: As we know that

$$\frac{d}{dx} F(x) = f(x) \rightarrow \text{Probability density}$$

$$f(x) = \begin{cases} 0 & \text{for} \quad x \le 0 \\ 1 & \text{for} \quad 0 < x < 1 \\ 0 & \text{for} \quad x \geq 1 \end{cases}$$

EXAMPLE

A random variable X has the following probability distribution function

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$$f(x) = \begin{cases} 2x & ; \ 0 < x < 1 \\ 0 & ; \ \text{otherwise} \end{cases}$$

Find (i)- $P(\frac{1}{4} \le X \le \frac{1}{2})$ (ii) $P(X > \frac{3}{4})$

(iii)- $P(-\frac{1}{2} < X < \frac{1}{2})$ (iv) $P(X = \frac{1}{2})$

Sol: A random variable X has the following probability distribution function

$$f(x) = \begin{cases} 2x & ; \ 0 < x < 1 \\ 0 & ; \ \text{otherwise} \end{cases}$$

(i)- $P(\frac{1}{4} \le x \le \frac{1}{2})$

$$P(\tfrac{1}{4} \le x \le \tfrac{1}{2}) = \int_{1/4}^{1/2} f(x)\,dx$$

$$= \int_{1/4}^{1/2} 2x\,dx = 2\int_{1/4}^{1/2} x\,dx = 2\left|\frac{x^2}{2}\right|_{1/4}^{1/2} = \left|x^2\right|_{1/4}^{1/2}$$

$$= \frac{1}{4} - \frac{1}{16}$$

$$P(\tfrac{1}{4} \le x \le \tfrac{1}{2}) = \frac{3}{16}$$

(ii)- $P(X > \frac{3}{4})$

$$P(X > \tfrac{3}{4}) = \int_{3/4}^{+\infty} f(x)\,dx$$

$$= \int_{3/4}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx$$

$$= \int_{3/4}^{1} 2x\,dx + \int_{1}^{\infty} 0\,dx$$

$$= 2\left|\frac{x^2}{2}\right|_{3/4}^{1} = 1 - \frac{9}{16} = \frac{16-9}{16} = \frac{7}{16}$$

$$P(X > \tfrac{3}{4}) = \frac{7}{16}$$

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(iii) - $P(-\frac{1}{2} < X < \frac{1}{2})$

$$P(-\frac{1}{2} < X < \frac{1}{2}) = \int_{-1/2}^{1/2} f(x)\,dx$$

$$= \int_{-1/2}^{0} f(x)\,dx + \int_{0}^{1/2} f(x)\,dx$$

$$= 0 + \int_{0}^{1/2} 2x\,dx$$

$$= 2\left|\frac{x^2}{2}\right|_0^{1/2}$$

$$P(-\frac{1}{2} < X < \frac{1}{2}) = \frac{1}{4}$$

(iv) - $P(X = \frac{1}{2})$

$$P(X = \frac{1}{2}) = 0$$

Because Area of one point does not exist

<u>DISTRIBUTION FUNCTION:</u>

$$F(x) = P(X \le x) = \int_0^x f(x)\,dx$$

$$= \int_0^x 2x\,dx = x^2$$

$$F(x) = x^2$$

(i) - $P(\frac{1}{4} < X < \frac{1}{2}) = F(\frac{1}{2}) - F(\frac{1}{4}) = (\frac{1}{2})^2 - (\frac{1}{4})^2$

$$= \int_0^{1/2} 2x\,dx - \int_0^{1/4} 2x\,dx$$

$$= \left|x^2\right|_0^{1/2} - \left|x^2\right|_0^{1/4}$$

$$= \frac{1}{4} - \frac{1}{16}$$

$$= \frac{3}{16}$$

(ii) - $P(X > \frac{3}{4}) = P(\frac{3}{4} < X < 1)$

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$$= F(1) - F\left(\frac{3}{4}\right)$$

$$= \int_0^1 2x\,dx - \int_0^{3/4} 2x\,dx$$

$$= \left|x^2\right|_0^1 - \left|x^2\right|_0^{3/4}$$

$$= 1 - \frac{9}{16} = \frac{7}{16}$$

EXAMPLE

Given the following p.d.f of a random variable X

$$f(x) = \begin{cases} A(4x - 2x^2) & ; 0 \le x \le 2 \\ 0 & ; \text{elsewhere} \end{cases}$$

Find the value of A.

Sol: As we know that

Total Area = 1

$$\int_0^2 A(4x - 2x^2)\,dx = 1$$

$$A\int_0^2 (4x - 2x^2)\,dx = 1$$

$$A\left|\frac{4x^2}{2} - \frac{2x^3}{3}\right|_0^2 = 1$$

$$A\left[8 - \frac{16}{3}\right] = 1$$

$$A\left(8/3\right) = 1$$

$$\boxed{A = 3/8}$$

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EXAMPLE

A continous random variable X has the following p.d.f

$$f(x) = \begin{cases} x/2 & ; \quad 0 \le x < 1 \\ \frac{1}{4}(3-x) & ; \quad 1 \le x < 2 \\ 1/4 & ; \quad 2 \le x < 3 \\ \frac{1}{4}(4-x) & ; \quad 3 \le x < 4 \\ 0 & \quad \text{elsewhere} \end{cases}$$

Compute

$P(X \geq 3)$, $P(X=2)$, $P(|X|<1.5)$, $P(1<X<3)$

Sol:

$$P(X \geq 3) = P(3 \le X < \infty)$$

$$= P(3 \le X \le 4) + P(X \geq 4)$$

$$= \int_3^4 \frac{1}{4}(4-x)\,dx + 0$$

$$= \frac{1}{4}\left|4x - \frac{x^2}{2}\right|_3^4$$

$$= \frac{1}{4}\left[4(4) - \frac{(4)^2}{3} - 4(3) + \frac{(3)^2}{2}\right]$$

$$= \frac{1}{4}\left[16 - \frac{16}{3} - 12 + \frac{9}{2}\right]$$

$$= \frac{1}{4}\left(\frac{1}{2}\right)$$

$$P(X \geq 3) = \frac{1}{8}$$

ii)- $P(X=2)$

$\therefore P(X=2) = 0$

$\because X$ is a continous random variable.

iii)- $P(|X|<1.5)$

$$= P(-1.5 < X < 1.5)$$

$$= P(-1.5 < X < 0) + P(0 < X < 1) + P(1 < X < 1.5)$$

$$= 0 + \int_0^1 \frac{x}{2}\,dx + \int_1^{1.5} \frac{1}{4}(3-x)\,dx$$

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$$= \left|\frac{x^2}{4}\right|_0^1 + \frac{1}{4}\left|3x - \frac{x^2}{2}\right|_1^{1.5}$$

$$= \frac{1}{4} + \frac{1}{4}\left[3(1.5) - \frac{(1.5)^2}{2} - 3(1) + \frac{1}{2}\right]$$

$$= \frac{1}{4} + \frac{1}{4}\left[2 - 1.125\right]$$

$$= 0.46875$$

(iv)- $P(1 < x < 3)$

$$= P(1 < X < 2) + P(2 < x < 3)$$

$$= \int_1^2 \frac{1}{4}(3-x)\,dx + \int_2^3 \frac{1}{4}\,dx$$

$$= \frac{1}{4}\left|3x - \frac{x^2}{2}\right|_1^2 + \frac{1}{4}\left|x\right|_2^3$$

$$= \frac{1}{4}\left[6 - \frac{4}{2} - 3 + \frac{1}{2}\right] + \frac{1}{4}\left[3 - 2\right]$$

$$= 0.875$$

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MULTIVARIATE DISTRIBUTION

2 variables $\longrightarrow$ Bivariates

3 variables $\longrightarrow$ Trivariates

More $\longrightarrow$ Multivariates

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* ——————————————————————— *

2011 Q 2(b) EXAMPLE # 3.12 (JOHN E FRUEND'S)

Two caplets are selected at random from a bottle containing three aspirin, two sedative and four lexative caplets. If X and Y are respectively the numbers of aspirin and sedative caplets included among the two caplets drawn from the bottle. Find the probabilities with all possible pairs of values of X and Y.

Sol:

$$n(S) = \binom{9}{2} = 36$$

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let X denote the number of aspirin and Y denote the number of sedative.

$$P(X=0, Y=0) = \frac{\binom{3}{0}\binom{2}{0}\binom{4}{2}}{\binom{9}{2}} = \frac{\frac{3!}{0!3!} \cdot \frac{2!}{0!2!} \cdot \frac{4!}{2!2!}}{36} = \frac{6}{36} = \frac{1}{6}$$

$$P(X=0, Y=1) = \frac{\binom{3}{0}\binom{2}{1}\binom{4}{1}}{\binom{9}{2}} = \frac{2}{9}$$

$$P(X=1, Y=0) = \frac{\binom{3}{1}\binom{2}{0}\binom{4}{1}}{\binom{9}{2}} = \frac{1}{3}$$

$$P(X=1, Y=1) = \frac{\binom{3}{1}\binom{2}{1}\binom{4}{0}}{\binom{9}{2}} = \frac{1}{6}$$

$$P(X=2, Y=0) = \frac{\binom{3}{2}\binom{2}{0}\binom{4}{0}}{\binom{9}{2}} = \frac{1}{12}$$

$$P(X=0, Y=2) = \frac{\binom{3}{0}\binom{2}{2}\binom{4}{0}}{\binom{9}{2}} = \frac{1}{36}$$

We will show all values in table

X →

Y ↓	0	1	2
0	1/6	1/3	1/12
1	2/9	1/6	
2	1/36		

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JOINT PROBABILITY DISTRIBUTION

If X and Y are discrete random variables, the function given by $f(x,y) = P(X=x, Y=y)$ for each pair of values (x,y) within the range of X and Y is called the Joint probability distribution of X and Y.

REMARK:

It fulfills the following properties

i)- $f(x,y) \geq 0$

ii)- $\sum_x \sum_y f(x,y) = 1$

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EXAMPLE # 3.13 (JOHN E FREUNDS)

Determine the value of K for which the function given by

$$f(x,y) = Kxy \quad \text{for } x=1,2,3 \;,\; y=1,2,3$$

can serve as a Joint probability distribution.

Sol:

As we know that

$$\sum_{x=1}^{3} \sum_{y=1}^{3} f(x,y) = 1$$

$$\sum_{x=1}^{3} \sum_{y=1}^{3} Kxy = 1$$

$$K \sum_{x=1}^{3} [x + 2x + 3x] = 1$$

$$K \sum_{x=1}^{3} [6x] = 1$$

$$6K[1+2+3] = 1$$

$$6K[6] = 1 \quad \Rightarrow \quad \boxed{K = \frac{1}{36}}$$

——————————————————

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JOINT DISTRIBUTION FUNCTION (Discrete case)

If X and Y are discrete random variables, the function is given by

$$F(x,y) = P(X \le x, Y \le y) \quad \text{for } -\infty < x < +\infty, -\infty < y < +\infty$$

$$= \sum_{t \le x} \sum_{s \le y} f(t,s)$$

is called Joint distribution function.

EXAMPLE # 3.14 (JOHN E FRUEND'S)

With refrence to Example # 3.12 find $F(1,1)$.

Sol:

$$F(1,1) = P(X \le 1, Y \le 1)$$

$$= f(0,0) + f(0,1) + f(1,0) + f(1,1)$$

$$= \frac{1}{6} + \frac{2}{9} + \frac{1}{3} + \frac{1}{6}$$

$$= \frac{16}{18}$$

$$F(1,1) = \frac{8}{9}$$

JOINT PROBABILITY DENSITY FUNCTION

A bivariate function with values $f(x,y)$ defined over the xy-plane is called Joint probability density function of continous random variables X and Y iff

$$P[(X,Y) \in A] = \iint_A f(x,y)\,dx\,dy$$

for any region A in xy-plane.

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THEOREM # 3.8:

A Joint probability density function satisfy the following conditions.

i)- $f(x,y) \geq 0$

ii)- $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dx\,dy = 1$

* ——————————————— *

EXAMPLE # 3.15 (JOHN E FRUNDS)

Given the joint probability density function

$$f(x,y) = \begin{cases} \frac{3}{5}x(y+x) & ; 0<x<1,\ 0<y<2 \\ 0 & ; \text{elsewhere} \end{cases}$$

of two random variables X and Y, find $P[(X,Y)\in A]$, where A is the region $\{(x,y) \mid 0<x<\frac{1}{2},\ 1<y<2\}$

Sol:

$$P[(X,Y)\in A] = P(0<X<\tfrac{1}{2},\ 1<Y<2)$$

$$= \int_1^2\int_0^{1/2} \frac{3}{5}x(y+x)\,dx\,dy$$

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$$= \frac{3}{5}\int_1^2\int_0^{1/2} (xy+x^2)\,dx\,dy$$

$$= \frac{3}{5}\int_1^2 \left|\frac{x^2y}{2} + \frac{x^3}{3}\right|_0^{1/2} dy$$

$$= \frac{3}{5}\int_1^2 \left[\frac{y}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\right)^3 - 0 - 0\right] dy$$

$$= \frac{3}{5}\int_1^2 \left(\frac{y}{8} + \frac{1}{24}\right) dy$$

$$= \frac{3}{5}\left|\frac{y^2}{8\cdot 2} + \frac{y}{24}\right|_1^2$$

$$= \frac{3}{5}\left[\frac{11}{48}\right] = \frac{11}{80} \text{ Ans}$$

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JOINT DISTRIBUTION FUNCTION (Continous case)

If X and Y are continous random variables, the function given by

$$F(x,y) = P(X \le x, Y \le y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(s,t)\,ds\,dt \;, -\infty < x < \infty, -\infty < y < +\infty$$

where $f(s,t)$ is the value of Joint probability density of X and Y at (s,t) is called Joint distribution function of X and Y.

EXAMPLE # 3.16 (JOHN E FRUEND'S)

If the joint probability density of X and Y is given by

$$f(x,y) = \begin{cases} x+y & \text{for } 0<x<1, 0<y<1 \\ 0 & \text{elsewhere} \end{cases}$$

find the Joint distribution function of these two variables.

Sol: If either $x<0$ or $y<0$

Then $f(x,y)=0$

So $F(x,y)=0$

Now,

For $0<x<1$, $0<y<1$

$$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(t,s)\,ds\,dt$$

$$= \int_{-\infty}^{x}\int_{-\infty}^{y} (t+s)\,ds\,dt$$

$$= \int_{0}^{x}\int_{0}^{y} (t+s)\,ds\,dt \qquad \left(\because \int_{-\infty}^{x} = \int_{-\infty}^{0} + \int_{0}^{x}\right)$$

$$= \int_{0}^{x} \left| ts + \frac{s^2}{2} \right|_0^y dt$$

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$$= \int_0^x \left(yt + \frac{y^2}{2}\right) dt$$

$$= \left| y \cdot \frac{t^2}{2} + \frac{y^2}{2} t \right|_0^x$$

$$= \frac{yx^2}{2} + \frac{y^2}{2} x$$

$$F(x,y) = \frac{1}{2} xy(x+y)$$

For $x > 1$ and $0 < y < 1$

$$F(x,y) = \int_0^1 \int_0^y (t+s)\, ds\, dt$$

$$= \int_0^1 \left| ts + \frac{s^2}{2} \right|_0^y dt$$

$$= \int_0^1 \left(ty + \frac{y^2}{2}\right) dt$$

$$= \left| y\frac{t^2}{2} + \frac{y^2}{2} t \right|_0^1$$

$$= \frac{y}{2} + \frac{y^2}{2}$$

$$F(x,y) = \frac{1}{2} y(y+1)$$

For $0 < x < 1$ and $y > 1$

$$F(x,y) = \int_0^x \int_0^1 (t+s)\, ds\, dt$$

$$= \int_0^x \left| ts + \frac{s^2}{2} \right|_0^1 dt$$

$$= \int_0^x \left(t + \frac{1}{2}\right) dt$$

$$= \left| \frac{t^2}{2} + \frac{1}{2} t \right|_0^x$$

$$= \frac{x^2}{2} + \frac{x}{2}$$

$$F(x,y) = \frac{1}{2} x(x+1)$$

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For $x>1$ and $y>1$

$$F(x,y) = \int_0^1 \int_0^1 (t+s)\, ds\, dt$$

$$= \int_0^1 \left| ts + \frac{s^2}{2} \right|_0^1 dt$$

$$= \int_0^1 \left(t + \frac{1}{2}\right) dt$$

$$= \left| \frac{t^2}{2} + \frac{1}{2} t \right|_0^1$$

$$= \frac{1}{2} + \frac{1}{2}$$

$$F(x,y) = 1$$

Since, Joint distribution function is

$$F(x,y) = \begin{cases} 0 & ; \quad x \le 0 \,,\ y \le 0 \\ \frac{1}{2} xy(x+y) & ; \quad 0<x<1 \,,\ 0<y<1 \\ \frac{1}{2} y(y+1) & ; \quad x>1 \,,\ 0<y<1 \\ \frac{1}{2} x(x+1) & ; \quad 0<x<1 \,,\ y>1 \\ 1 & ; \quad x>1 \,,\ y>1 \end{cases}$$

EXAMPLE # 3.17 (JOHN E FRUEND'S)

Find the Joint probability density of the two random variables X and Y whose Joint distribution is given by

$$F(x,y) = \begin{cases} (1-e^{-x})(1-e^{-y}) & \text{for } x>0 \,,\ y>0 \\ 0 & ; \text{ elsewhere} \end{cases}$$

Also use the Joint probability density to determine $P(1<x<3, 1<y<2)$

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Sol: As $\frac{\partial^2 F(x,y)}{\partial x \partial y} = f(x,y)$

$$F(x,y) = (1-e^{-x})(1-e^{-y})$$
$$= 1 - e^{-y} - e^{-x} + e^{-x}e^{-y}$$
$$= 1 - e^{-y} - e^{-x} + e^{-(x+y)}$$

$$\frac{\partial^2}{\partial x \partial y} F(x,y) = \frac{\partial^2}{\partial x \partial y}\left[1 - e^{-y} - e^{-x} + e^{-(x+y)}\right]$$

$$= \frac{\partial}{\partial x}\left[0 - e^{-y}(-1) - 0 + e^{-(x+y)}(-1)\right]$$

$$= \frac{\partial}{\partial x}\left[e^{-y} - e^{-(x+y)}\right]$$

$$= 0 - e^{-(x+y)}(-1)$$

$$\frac{\partial^2 F(x,y)}{\partial x \partial y} = e^{-(x+y)}$$

So, the probability density is

$$f(x,y) = \begin{cases} e^{-(x+y)} & ; x>0, y>0 \\ 0 & \text{elsewhere} \end{cases}$$

Now,

$$P(1<X<3, 1<Y<2) = \int_1^2\int_1^3 e^{-(x+y)}\,dx\,dy$$

$$= \int_1^2\int_1^3 e^{-x}\cdot e^{-y}\,dx\,dy$$

$$= \int_1^2 e^{-y}\left|\frac{e^{-x}}{-1}\right|_1^3 dy$$

$$= -\int_1^2 e^{-y}\left[e^{-3} - e^{-1}\right]dy$$

$$= (e^{-1} - e^{-3})\int_1^2 e^{-y}\,dy$$

$$= (e^{-1} - e^{-3})\left|\frac{e^{-y}}{-1}\right|_1^2$$

$$= (e^{-1} - e^{-3})(e^{-1} - e^{-2})$$

$$= 0.074$$

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EXAMPLE # 3.18 (JOHN E FRUEND'S)

If the Joint probability distribution of three discrete random variables X, Y and Z is given by

$$f(x,y,z) = \frac{(x+y)z}{63} \quad \text{for } x=1,2,\ y=1,2,\ z=1,2$$

Find $P(X=2, Y+Z \le 3)$.

Sol:

$$P(X=2, Y+Z \le 3) = f(2,1,1) + f(2,1,2) + f(2,2,1)$$

$$= \frac{(2+1)1}{63} + \frac{(2+1)2}{63} + \frac{(2+2)1}{63}$$

$$= \frac{3}{63} + \frac{6}{63} + \frac{4}{63}$$

$$= \frac{13}{63} \text{ Ans}$$

EXAMPLE # 3.19 (JOHN E FRUEND'S)

If the trivariate probability density of X_1, X_2 and X_3 is given by

$$f(x_1, x_2, x_3) = \begin{cases} (x_1+x_2)e^{-x_3} & \text{for } 0<x_1<1,\ 0<x_2<1,\ x_3>0 \\ 0 & \text{elsewhere} \end{cases}$$

Find $P[(X_1, X_2, X_3) \in A]$ where A is the region

$$\{(x_1, x_2, x_3) \mid 0 < x_1 < \tfrac{1}{2},\ \tfrac{1}{2} < x_2 < 1,\ x_3 < 1\}$$

Sol:

$$P[(X_1, X_2, X_3) \in A] = P(0 < X_1 < \tfrac{1}{2},\ \tfrac{1}{2} < X_2 < 1,\ X_3 < 1)$$

$$= \int_0^1 \int_{1/2}^1 \int_0^{1/2} (x_1 + x_2)e^{-x_3}\, dx_1\, dx_2\, dx_3$$

$$= \int_0^1 \int_{1/2}^1 \left[\left(\frac{x_1^2}{2} + x_2 x_1\right) e^{-x_3}\right]_0^{1/2} dx_2\, dx_3$$

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$$= \int_0^1 \int_{1/2}^1 \left(e^{-x_3} \cdot \frac{1}{8} + \frac{x_2}{2} \cdot e^{-x_3}\right) dx_2\, dx_3$$

$$= \int_0^1 \left| \frac{e^{-x_3}}{8} \cdot x_2 + \frac{x_2^2}{2} \cdot \frac{e^{-x_3}}{2} \right|_{1/2}^1 dx_3$$

$$= \int_0^1 \left[\frac{e^{-x_3}}{8} + \frac{e^{-x_3}}{4} - \frac{e^{-x_3}}{16} - \frac{e^{-x_3}}{16} \right] dx_3$$

$$= \int_0^1 \frac{e^{-x_3}}{4}\, dx_3$$

$$= \frac{1}{4} \left| \frac{e^{-x_3}}{-1} \right|_0^1$$

$$= \frac{1}{4}(1 - e^{-1})$$

$$= 0.158 \text{ Ans.}$$

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MARGINAL DISTRIBUTIONS

i) **Discrete:**

If X and Y are discrete random variables then marginal distribution of X at x is given by

$$h(x) = \sum_y f(x, y)$$

If X and Y are disrete random variables then marginal distribution of Y at y is given by

$$g(y) = \sum_x f(x, y)$$

ii). **Continous:**

If X and Y are continous random variables Then marginal density is given by

$$h(x) = \int_{-\infty}^{+\infty} f(x, y)\, dy \quad \text{for } -\infty < x < +\infty$$

If X and Y are continous random variables then marginal density is given by

$$g(y) = \int_{-\infty}^{+\infty} f(x, y)\, dx \quad \text{for } -\infty < y < +\infty$$

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EXAMPLE # 3.20 (JOHN E FREUND'S)

In Example 3.12, we derived the joint probability distribution of random variables X and Y, the number of asprin caplets and number of sedative caplets included among two caplets drawn at random from bottle containing three aspirin, two sedative and four laxative caplets. Find the probability distribution of X alone and Y alone.

Sol:

The results of Example 3.12 is given below

Y ↓ \ X →	0	1	2
0	1/6	1/3	1/12
1	2/9	1/6	
2	1/36		

Now we will find marginal distribution of X.

$$f(x) = \sum_{y} f(x,y) \quad ,$$

$$f(0) = \sum_{y=0}^{2} f(0,y)$$

$$= f(0,0) + f(0,1) + f(0,2)$$

$$= \frac{1}{6} + \frac{2}{9} + \frac{1}{36} = \frac{15}{36}$$

$$f(1) = \sum_{y=0}^{2} f(1,y)$$

$$= f(1,0) + f(1,1)$$

$$= \frac{1}{3} + \frac{1}{6} + 0 = \frac{3}{6} = \frac{1}{2}$$

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$$g(2) = \sum_{y=0}^{2} f(x,y)$$

$$= f(2,0) = \frac{1}{2}$$

Now, we will find the marginal distribution of y.

$$g(y) = \sum_{x} f(x,y)$$

$$g(0) = \sum_{x=0}^{2} f(x,0)$$

$$= f(0,0) + f(1,0) + f(2,0)$$

$$= \frac{1}{6} + \frac{1}{3} + \frac{1}{12} = \frac{7}{12}$$

$$g(1) = \sum_{x=0}^{2} f(x,1)$$

$$= f(0,1) + f(1,1)$$

$$= \frac{2}{9} + \frac{1}{6} = \frac{7}{18}$$

$$g(2) = \sum_{x=0}^{2} f(x,2)$$

$$= f(0,2) = 1/36$$

Hence

$y \backslash x$	0	1	2	$g(y)$
0	$1/6$	$1/3$	$1/12$	$7/12$
1	$\frac{2}{9}$	$\frac{1}{6}$		$7/18$
2	$\frac{1}{36}$			$1/36$
$g(x)$	$15/36$	$1/2$	$1/12$	

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FOR VERIFICATION

$$g(x) = \frac{15}{36} + \frac{1}{2} + \frac{1}{12} = \frac{15+18+3}{36} = \frac{36}{36} = 1$$

$$h(y) = \frac{7}{12} + \frac{7}{18} + \frac{1}{36} = \frac{21+14+1}{36} = \frac{36}{36} = 1$$

نوٹ:- جب ہم marginal distribution نکالنا چاہتے ہوں تو x کی قیمت کیلئے ہم اسکی تمام values کو fix کرکے y کی تمام values کو add کر دیں گے۔
جب ہم y کی M.D نکالنا چاہتے ہوں تو ہم y کی values کو باری باری fix کرکے x کی تمام values کو add کر دیں گے۔

آخر میں اپنے سوال کو check کرنے کیلئے M.D جو کہ x اور y کی ہیں ان کو باری باری جمع کرنے سے اگر جواب 1 آجائے تو ہمارا سوال ٹھیک ہے وگرنہ ہمارا سوال غلط ہوگا

EXAMPLE # 3.21 (JOHN E FRUENDS)

Given the Joint probability density

$$f(x,y) = \begin{cases} \frac{2}{3}(x+2y) & \text{for } 0<x<1,\ 0<y<1 \\ 0 & \text{elsewhere} \end{cases}$$

for the marginal densities of x and y.

Sol:

First we find the marginal density of x.

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \quad , \text{ for } -\infty<x<+\infty$$

$$= \int_0^1 \frac{2}{3}(x+2y)\,dy$$

$$= \frac{2}{3}\left| xy + \frac{2y^2}{2} \right|_0^1$$

$$= \frac{2}{3}\left| xy + y^2 \right|_0^1$$

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$$g(x) = \frac{2}{3}[x+1]$$

Now, we find the marginal density of Y

$$h(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx \quad , \quad -\infty < y < +\infty$$

$$= \int_0^1 \frac{2}{3}(x+2y)\,dx$$

$$= \frac{2}{3}\left| \frac{x^2}{2} + 2xy \right|_0^1$$

$$= \frac{2}{3}\left[\frac{1}{2} + 2y \right]$$

$$= \frac{1}{3}(1+4y)$$

EXAMPLE # 3.22 (JOHN E FRUEND'S)

Considering again the trivariate probability density of Ex # 3.19

$$f(x_1,x_2,x_3) = \begin{cases} (x_1+x_2)e^{-x_3} & \text{for } 0<x_1<1,\ 0<x_2<1,\ x_3>1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the Joint marginal density of X_1 and X_3 and the marginal density of X_1 alone?

Sol:

First we find the Joint marginal density of X_1 and X_3.

$$g(x_1,x_3) = \int_{-\infty}^{+\infty} f(x_1,x_2,x_3)\,dx_2$$

$$= \int_0^1 (x_1+x_2)e^{-x_3}\,dx_2$$

$$= e^{-x_3}\left| x_1x_2 + \frac{x_2^2}{2} \right|_0^1$$

$$= e^{-x_3}\left[x_1 + \frac{1}{2} \right]$$

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Now, we find the marginal ~~distribution~~ density at x_1.

$$g(x_1) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x_1, x_2, x_3)\, dx_3\, dx_2$$

$$= \int_0^1 \int_0^\infty (x_1 + x_2) e^{-x_3}\, dx_3\, dx_2$$

$$= \int_0^1 (x_1 + x_2) \left[\int_0^\infty e^{-x_3}\, dx_3 \right] dx_2$$

$$= \int_0^1 (x_1 + x_2) \left| \frac{e^{-x_3}}{-1} \right|_0^\infty dx_2$$

$$= \int_0^1 (x_1 + x_2) \left[\frac{e^{-\infty}}{-1} - \frac{e^0}{-1} \right] dx_2$$

$$= \int_0^1 (x_1 + x_2) [0 + 1]\, dx_2$$

$$= \left| x_1 x_2 + \frac{x_2^2}{2} \right|_0^1$$

$$h(x_1) = x_1 + \frac{1}{2} \quad \underline{\underline{\text{Ans}}}$$

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CONDITIONAL DISTRIBUTIONS

As we know that conditional probability of event A, given event B as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Suppose that A and B are the events $X=x$ and $Y=y$ so we can write

$$P(X=x|Y=y) = \frac{P(X=x, Y=y) \longrightarrow \text{joint}}{P(Y=y) \longrightarrow \text{marginal}}$$

$$\text{conditional} = \frac{\text{Joint}}{\text{marginal}}$$

$$f(x|y) = \frac{f(x,y)}{h(y)}$$

$$f(y|x) = \frac{f(x,y)}{g(x)}$$

When X and Y are continous random variables, the probability distribution is replaced by probabality density.

EXAMPLE # 3.23 (JOHN E FRUEND'S)

With the refrence of Example #3.20, find the conditional ~~probability~~ distribution of X given $Y=1$.

Sol:

Results of Example 3.20 is shown as follows

Y \ X	0	1	2	h(y)
0	1/6	1/3	1/12	7/12
1	2/9	1/6		7/18
2	1/36			1/36
g(x)	15/12	1/2	1/12	

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As we know that

$$f(x|y) = \frac{f(x,y)}{h(y)}$$

$$f(x|1) = \frac{f(x,1)}{h(1)} \quad , x = 0, 1, 2$$

$$f(0|1) = \frac{f(0,1)}{h(1)} = \frac{2/9}{7/18} = \frac{4}{7}$$

$$f(1|1) = \frac{f(1,1)}{h(1)} = \frac{1/6}{7/18} = \frac{3}{7}$$

$$f(2|1) = \frac{f(2,1)}{h(1)} = \frac{0}{7/18} = 0$$

Example # 3.24 (JOHN E FRUEND'S)

Given that Joint Probability density

$$f(x,y) = \begin{cases} \frac{2}{3}(x+2y) & \text{for } 0<x<1, 0<y<1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the conditional density of X given Y=y and use it to evaluate $P(X \leq \frac{1}{2} | Y = \frac{1}{2})$

Sol:

we will find the conditional density of X given Y=y.

$$P(X|Y=y) = \frac{f(x,y)}{h(y)} \quad \text{---} (1)$$

$$h(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx$$

$$= \int_{0}^{1} \frac{2}{3}(x+2y)\,dx$$

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$$= \frac{2}{3}\left[\frac{x^2}{2} + 2xy\right]_0^1$$

$$= \frac{2}{3}\left[\frac{1}{2} + 2y\right] = \frac{2}{3}\left[\frac{1+4y}{2}\right] = \frac{1}{3}[1+4y]$$

putting in (ii), we get

$$P(X|Y=y) = \frac{\frac{2}{3}(x+2y)}{\frac{1}{3}(1+4y)} = 2\frac{(x+2y)}{(1+4y)} = \frac{2x+4y}{1+4y}$$

Now, we find the conditional density of X given $Y=y$

$$P(X|\tfrac{1}{2}) = \frac{2x+4(\frac{1}{2})}{1+4(\frac{1}{2})} = \frac{2x+2}{1+2} = \frac{2}{3}(x+1)$$

Now, we evaluate

$$P(X \le \tfrac{1}{2} | Y = \tfrac{1}{2}) = \int_0^{1/2} \frac{2}{3}(x+1)\,dx$$

$$= \frac{2}{3}\left|\frac{x^2}{2} + x\right|_0^{1/2}$$

$$= \frac{2}{3}\left[\frac{1}{8} + \frac{1}{2}\right]$$

$$= \frac{5}{12} \text{ Ans}$$

* ——————————————————— *

EXAMPLE # 3.25 (JOHN E FRUEND'S)

Given joint probability density

$$f(x,y) = \begin{cases} 4xy & \text{for } 0<x<1,\ 0<y<1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the marginal densities of X & Y and the conditional density of X given $Y=y$.

Sol:

First, we find the marginal density of X.

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$$g(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy$$

$$= \int_0^1 4xy\,dy = 4x\left|\frac{y^2}{2}\right|_0^1 = 4x\left[\frac{1}{2}-0\right] = 4x\left(\frac{1}{2}\right)$$

$$g(x) = 2x \text{ Ans}$$

Now, we find the marginal density of Y.

$$h(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx$$

$$= \int_0^1 4xy\,dx = 4y\left|\frac{x^2}{2}\right|_0^1 = 4y\left[\frac{1}{2}-0\right] = 4y\left(\frac{1}{2}\right)$$

$$h(y) = 2y \text{ Ans}$$

Now, we find the conditional probability of X given $Y = y$.

$$f(x|y) = \frac{f(x,y)}{h(y)}$$

$$= \frac{4xy}{2y} = 2x \text{ Ans}$$

INDEPENDENT RANDOM VARIABLES

As,

$$P(A \cap B) = P(A) \cdot P(B)$$

$X_1, X_2 \rightarrow$ independent random variable

$$f(x_1, x_2, \ldots x_n) = f_1(x_1) \cdot f_2(x_2) \cdots\cdots f_n(x_n)$$

EXAMPLE # 3.27 (JOHN E FRUEND'S)

Given the independent random variables X_1, X_2 and X_3 with the probability densities

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$$f_1(x_1) = \begin{cases} e^{-x_1} & \text{for } x_1 > 0 \\ 0 & \text{elsewhere} \end{cases}, \quad f_2(x_2) = \begin{cases} 2e^{-2x_2} & \text{for } x_2 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$f_3(x_3) = \begin{cases} 3e^{-3x_3} & \text{for } x_3 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find their joint probability density and use it to evaluate the probability $P(X_1 + X_2 \le 1, X_3 > 1)$.

Sol:

As we know that

$$f(x_1, x_2, x_3) = f_1(x_1) \cdot f_2(x_2) \cdot f_3(x_3)$$
$$= e^{-x_1} \cdot 2e^{-2x_2} \cdot 3e^{-3x_3}$$
$$= 6e^{-x_1 - 2x_2 - 3x_3}$$

For $x_1 > 0, x_2 > 0, x_3 > 0$ and $f(x_1, x_2, x_3) = 0$ elsewhere

Thus

$$P(X_1 + X_2 \le 1, X_3 > 1) = \int_1^{\infty} \int_0^1 \int_0^{1-x_2} 6e^{-x_1 - 2x_2 - 3x_3} \, dx_1 \, dx_2 \, dx_3$$

$$= (1 - 2e^{-1} + e^{-2}) e^{-3}$$
$$= 0.020$$

$\left(\because X_1 + X_2 \le 1 \right.$
$X_1 = 1 - X_2$
$\left. \text{so } 0 \to 1 - X_2 \right.$

EXAMPLE #7.6 (SHER CHOUDRY)

S/2018

An Urn contains 3 black, 2 red and 3 green balls and 2 balls are selected at random from it. if X is the number of black balls and Y is the number of red balls selected, then find

i) The joint probability function $f(x,y)$ (ii) $P(X=1, Y \le 1)$

(iii) $P(X+Y \le 1)$

iv) The marginal p.d. $g(x)$ and $h(y)$

v) The condition p.d. $f(x|1)$

(ii) $P(X+Y>1) = ?$

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vi). $P(X=0 \mid Y=1)$ and

vii). Are X and Y independent?

Sol:

Black → 3
Red → 2
Green → 3

$n(S) = {}^8C_2 = 28$

X : no. of black balls = 0, 1, 2

Y : no. of red balls = 0, 1, 2

(i)

$$f(x,y) = P(X=x, Y=y) \qquad \left(\because P(A) = \frac{n(A)}{n(S)}\right)$$

$$= \frac{{}^3C_x \, {}^2C_y \, {}^3C_{2-x-y}}{{}^8C_2} \qquad x=0,1,2,\ y=0,1,2$$

$$f(0,0) = \frac{{}^3C_0 \, {}^2C_0 \, {}^3C_2}{{}^8C_2} = \frac{3}{28}$$

$$f(0,1) = \frac{{}^3C_0 \, {}^2C_1 \, {}^3C_1}{{}^8C_2} = \frac{6}{28}$$

$$f(0,2) = \frac{{}^3C_0 \, {}^2C_2 \, {}^3C_0}{{}^8C_2} = \frac{1}{28}$$

$$f(1,0) = \frac{{}^3C_1 \, {}^2C_0 \, {}^3C_1}{{}^8C_2} = \frac{9}{28}$$

$$f(1,1) = \frac{{}^3C_1 \, {}^2C_1 \, {}^3C_0}{{}^8C_2} = \frac{6}{28}$$

$$f(2,0) = \frac{{}^3C_2 \, {}^2C_0 \, {}^3C_0}{{}^8C_2} = \frac{3}{28}$$

$$f(2,1) = \frac{{}^3C_2 \, {}^2C_1 \, {}^3C_{-1}}{{}^8C_2} \quad \text{impossible event}$$

$$f(1,2) = \frac{{}^3C_1 \, {}^2C_2 \, {}^3C_{-1}}{{}^8C_2} \quad \text{impossible event}$$

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$$f(2,2) = \frac{{}^3C_2 \, {}^2C_2 \, {}^3C_{-2}}{{}^8C_2} \quad \text{impossible event}$$

$y \backslash x$	0	1	2	$h(y)$
0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
1	$\frac{6}{28}$	$\frac{6}{28}$		$\frac{12}{28}$
2	$\frac{1}{28}$			$\frac{1}{28}$
$g(x)$	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$	

-(ii)-

$$P(X=1, Y \le 1) = P(X=1, Y=0) + P(X=1, Y=1)$$
$$= f(1,0) + f(1,1)$$
$$= \frac{9}{28} + \frac{6}{28}$$
$$= \frac{15}{28}$$

-(iii)-

$$P(X+Y \le 1) = f(0,0) + f(0,1) + f(1,0)$$
$$= \frac{3}{28} + \frac{6}{28} + \frac{9}{28}$$
$$= \frac{9}{14}$$

(iv)

Marginal distribution of x

x	0	1	2
$g(x)$	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$

$\left(\because \frac{10}{28} + \frac{15}{28} + \frac{3}{28} = 1 \right)$

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Marginal distribution of y

y	0	1	2
h(y)	$\frac{15}{28}$	$\frac{12}{28}$	$\frac{1}{28}$

(v)

$$f(x|y) = \frac{f(x,y)}{h(y)}$$

$$f(x|y=1) = \frac{f(x,1)}{h(1)}, \quad x=0,1,2$$

$$f(0|y=1) = \frac{f(0,1)}{h(1)} = \frac{6/28}{12/28} = \frac{1}{2}$$

$$f(x=1|y=1) = \frac{f(1,1)}{h(1)} = \frac{6/28}{12/28} = \frac{1}{2}$$

$$f(x=2|y=1) = \frac{f(2,1)}{h(1)} = 0$$

(vi)

$$P(X=0|Y=1) = \frac{P(X=0, Y=1)}{P(Y=1)}$$

$$= \frac{f(0,1)}{h(1)}$$

$$= \frac{1}{2}$$

(vii)

As $f(0,0) = \frac{3}{28}$, $g(0) = \frac{10}{28}$, $h(0) = \frac{15}{28}$

$$f(0,0) \neq g(0)h(0)$$

which shows that X and Y are not independent.

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EXAMPLE

Suppose that joint p.d.f of (x,y) is given by

$$f(x,y) = \begin{cases} x^2 + \frac{1}{3}xy & , 0 \le x \le 1 , 0 \le y \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

a) check if $f(x,y)$ is a p.d.f

b) Compute

i) $P(X > \frac{1}{2})$ (ii) $P(Y < X)$ (iii) $P(X+Y > 1)$ (iv) $P(Y < \frac{1}{2} \mid X < \frac{1}{2})$

Sol:

(a)

we can easily observe that $f(x,y) \geq 0$ for given values of x and y.

Now we check the total area is unity.

$$\text{Total area} = \int_0^1 \int_0^2 (x^2 + \frac{1}{3}xy)\, dx\, dy$$

$$= \int_0^1 \left| x^2 y + \frac{xy^2}{6} \right|_0^2 dx$$

$$= \int_0^1 (2x^2 + \frac{2}{3}x)\, dx$$

$$= \left| \frac{2x^3}{3} + \frac{x^2}{3} \right|_0^1$$

$$= \frac{2}{3} + \frac{1}{3}$$

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Total area $= 1$

(b)

First we check the marginal p.d.f of x.

$$g(x) = \int_y f(x,y)\, dy$$

$$= \int_0^2 (x^2 + \frac{1}{3}xy)\, dy$$

$$= \left| x^2 y + \frac{xy^2}{6} \right|_0^2$$

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$$g(x) = 2x^2 + \frac{2}{3}x \quad , \quad 0 \le x \le 1$$

(i)

$$P(X > \tfrac{1}{2}) = \int_{1/2}^{1} g(x)\,dx = \int_{1/2}^{1} \left(2x^2 + \frac{2}{3}x\right) dx = \frac{5}{6}$$

(ii)

$$P(Y < X) = P(X > Y) = \int_{y}^{1} \left(2x^2 + \frac{2}{3}x\right) dx = 1 - \left(\frac{2y^3}{3} + y^2\right)$$

(iii)

$$P(X+Y > 1) = P(X > 1-Y) = \int_{0}^{2}\int_{1-y}^{1} \left(x^2 + \frac{1}{3}xy\right) dx\,dy = ?\ \text{solve it}$$

(iv)

$$P(Y < \tfrac{1}{2} \mid X < \tfrac{1}{2}) = \frac{P(X < \frac{1}{2}, Y < \frac{1}{2})}{P(X < \frac{1}{2})} \quad \text{——— (A)}$$

$$P(X < \tfrac{1}{2}, Y < \tfrac{1}{2}) = \int_{0}^{1/2}\int_{0}^{1/2} f(x,y)\,dx\,dy$$

$$= \int_{0}^{1/2}\int_{0}^{1/2} \left(x^2 + \frac{1}{3}xy\right) dx\,dy$$

$$= \frac{1}{32} + \frac{1}{576}$$

$$P(X < \tfrac{1}{2}) = \int_{0}^{1/2} g(x)\,dx = \frac{1}{6}$$

Putting in (A)

$$P(Y < \tfrac{1}{2} \mid X < \tfrac{1}{2}) = \frac{\frac{1}{32} + \frac{1}{576}}{1/6} = \text{———}$$ ANS

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CONTINOUS PROBABILITY DISTRIBUTION

If x is a continous random variable with a probability function $f(x)$ then we call this function a "probability density function" (p.d.f) if

i)- $f(x) \geq 0 \quad \forall x$

ii)- $\int_{-\infty}^{+\infty} f(x)dx = 1$

(01)- THE UNIFORM DISTRIBUTION

A random variable has a uniform distribution and it is referred to as a continous uniform random variable if and only if its probability density is given by

$$f(x) = \begin{cases} \frac{1}{b-a} & , a < x < b \\ 0 & , \text{elsewhere} \end{cases}$$

i) Total Area = ?

$$\text{Total Area} = \int_a^b f(x)\,dx$$

$$= \int_a^b \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{b-a}\left|x\right|_a^b$$

$$= \frac{1}{b-a}(b-a)$$

Total area = 1

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ii)- Mean = ?

$$\text{Mean} = E(X) = \int_x x f(x)$$

$$= \int_a^b x \cdot \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \left| \frac{x^2}{2} \right|_a^b$$

$$= \frac{1}{2(b-a)} (b^2 - a^2)$$

$$= \frac{(b-a)(b+a)}{2(b-a)}$$

$$\boxed{\text{Mean} = \frac{b+a}{2}}$$

iii)- Variance = ?

$$\text{Variance} = E(X^2) - (E(X))^2 \quad \text{--- (1)}$$

$$E(X^2) = \int_x x^2 \cdot f(x)$$

$$= \int_a^b x^2 \cdot \frac{1}{b-a}\, dx = \frac{1}{b-a} \left| \frac{x^3}{3} \right|_a^b = \frac{b^3 - a^3}{3(b-a)}$$

$$= \frac{(b-a)(b^2+a^2+ab)}{3(b-a)}$$

$$E(X^2) = \frac{a^2+b^2+ab}{3}$$

putting all values in (1), we get

$$\text{variance} = \frac{a^2+b^2+ab}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{4(a^2+b^2+ab) - 3(a^2+b^2+2ab)}{12}$$

$$= \frac{a^2+b^2-2ab}{12}$$

$$\boxed{\text{variance} = \frac{(a-b)^2}{12}}$$

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iv) Moment Generating Function?

$$M_x(t) = E(e^{tx})$$

$$= \int_a^b e^{tx} \cdot \frac{1}{b-a} dx$$

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$$= \frac{1}{b-a} \left| \frac{e^{tx}}{t} \right|_a^b$$

$$M_x(t) = \frac{1}{t(b-a)} (e^{tb} - e^{ta})$$

---

(02)- THE EXPONENTIAL DISTRIBUTION

A random variable x has an exponential distribution and it is referred to as an exponential random variable if and only if its probability density is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , x > 0 \\ 0 & , \text{elsewhere} \end{cases}$$

where $\lambda > 0$, we can write $x \sim exp(\lambda)$

Sometimes we write

$$f(x) = \frac{1}{\beta} e^{-x/\beta}$$

i) Total Area = ?

$$\text{Total Area} = \int_x f(x)\,dx = \int_0^\infty \lambda e^{-\lambda x} dx = \lambda \left| \frac{e^{-\lambda x}}{-\lambda} \right|_0^\infty$$

$$= -\left| e^{-\lambda x} \right|_0^\infty = -[0-1]$$

Total Area = 1

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ii) - Mean = ?

$$\text{Mean} = E(X) = \int_x x f(x) = \int_0^\infty x \cdot \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_0^\infty x e^{-\lambda x}\, dx$$

$$= \lambda \left[x \cdot \frac{e^{-\lambda x}}{-\lambda} \Big|_0^\infty - \int_0^\infty \frac{e^{-\lambda x}}{-\lambda} \cdot 1\, dx \right]$$

$$= \frac{-\lambda}{\lambda} \left| x e^{-\lambda x} \right|_0^\infty + \frac{\lambda}{\lambda} \int_0^\infty e^{-\lambda x}\, dx$$

$$= 0 + \frac{e^{-\lambda x}}{-\lambda} \Big|_0^\infty$$

$$= -\frac{1}{\lambda} \left[e^{-\lambda x} \right|_0^\infty$$

$$= -\frac{1}{\lambda} [0 - 1]$$

A/2008
A/2012

$$\boxed{\text{Mean} = \frac{1}{\lambda}}$$

iii) - Variance = ?

$$\text{variance} = E(X^2) - (E(X))^2 \quad \text{--- (1)}$$

$$E(X^2) = \int_0^\infty x^2 \cdot \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_0^\infty x^2 \cdot e^{-\lambda x}\, dx$$

put $\lambda x = y$

$$dx = \frac{dy}{\lambda}$$

$$E(X^2) = \lambda \int_0^\infty \frac{y^2}{\lambda^2} \cdot e^{-y} \cdot \frac{dy}{\lambda}$$

$$= \frac{1}{\lambda^2} \int_0^\infty y^2 \cdot e^{-y}\, dy$$

"OR"

$$\lambda \left[\cancel{x^2 \cdot \frac{e^{-\lambda x}}{-\lambda} \Big|_0^\infty}^{\,0} - \int_0^\infty \frac{e^{-\lambda x}}{-\lambda} \cdot 2x \right]$$

$$= \lambda(0) + \frac{2\lambda}{\lambda} \int_0^\infty x e^{-\lambda x}\, dx$$

$$= 2 \left[\cancel{x \cdot \frac{e^{-\lambda x}}{-\lambda} \Big|_0^\infty}^{\,0} - \int_0^\infty \frac{e^{-\lambda x}}{-\lambda} \cdot dx \right]$$

$$= 2(0) + \frac{2}{\lambda} \int_0^\infty e^{-\lambda x}\, dx$$

$$= \frac{2}{\lambda} \left(\frac{1}{\lambda} \right) = \frac{2}{\lambda^2}$$

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$$= \frac{1}{\lambda^2}\Gamma 3 \qquad \left(\because \int_0^\infty e^{-x} x^{m-1} dx = \Gamma m\right)$$

$$E(x^2) = \frac{2!}{\lambda^2}$$

putting in (1), we get

$$\text{variance} = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

A/2012 $\boxed{\text{variance} = \frac{1}{\lambda^2}}$

iv). Moment Generating Function = ?

$$M_x(t) = E(e^{tx})$$

$$= \int_x e^{tx} \cdot \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty e^{(t-\lambda)x} dx$$

$$= \lambda \left. \frac{e^{(t-\lambda)x}}{(t-\lambda)} \right|_0^\infty$$

$$= \frac{\lambda}{t-\lambda} \left| e^{(t-\lambda)x} \right|_0^\infty$$

$$= \frac{\lambda}{t-\lambda} \left| e^{-(\lambda-t)x} \right|_0^\infty$$

$$= \frac{\lambda}{t-\lambda} [0-1]$$

$$\boxed{M_x(t) = \frac{-\lambda}{t-\lambda}}$$

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EXAMPLE

The distribution of long distance telephone call found to be exponential with mean duration of 3 minutes. Find the probability that a call will last

i)- More than 3 minute.

ii)- less than 5 minute.

iii)- b/w 1 and 4 minute.

Sol:

let X be the random variable for the length of long distance telephone calls duration in minutes.

i.e $X \sim exp(\lambda)$

As given that

$$Mean = 3$$

For exponential distribution, $Mean = \frac{1}{\lambda}$

$$\Rightarrow \lambda = \frac{1}{3}$$

So, $f(x) = \begin{cases} \frac{1}{3} e^{-x/3} & , x > 0 \\ 0 & \text{elsewhere} \end{cases}$

i)- $P(X > 3) = \int_3^\infty f(x)dx = \int_3^\infty \frac{1}{3} e^{-x/3} = -\left| e^{-x/3} \right|_3^\infty = -(0 - e^{-1}) = \frac{1}{e}$ Ans

ii)- $P(X < 5) = \int_0^5 f(x) = -\left| e^{-x/3} \right|_0^5 = -\left[e^{-5/3} - 1 \right] = 0.811$ Ans

iii)- $P(1 < X < 4) = \int_1^4 f(x)dx = -\left| e^{-x/3} \right|_1^4 = -\left[e^{-4/3} - e^{-1/3} \right] = 0.452$ Ans

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Binomial Distribution.

Formula:

$$P(X=x) = \binom{n}{x} p^x q^{n-x} \quad ; \; x = 0, 1, 2, \cdots n$$

It is also denoted by $b(x; n, p)$ where n and p are parameters.

Mean.

$$\mu = E(X) = \sum_{x=0}^{n} x \, P(X=x)$$

2011/2012

$$= \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x} \quad ; x = 0, 1, \cdots n$$

$$= \sum_{x=0}^{n} x \frac{n!}{x!\,(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \cancel{x} \frac{n!}{\cancel{x}(x-1)!\,(n-x)!} p^x q^{n-x}$$

$$= n \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!\,(n-x)!} p^x q^{n-x}$$

$$= n \sum_{x=1}^{n} \binom{n-1}{x-1} p^x q^{n-x}$$

Let $n-1 = m$ and $x-1 = y$

$\Rightarrow n = m+1$ and $x = 1+y$

$x = 1 \Rightarrow y = 0$

$x \to n \Rightarrow y_n \to m$

Then $\mu = np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} q^{n-x}$

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$$= np \sum_{y=0}^{m} \binom{m}{y} p^{y} q^{m-y}$$

$m-y$
$\Rightarrow n-x-1+1$
$(n-1)-(x-1)$

$$= np(1)$$

$$\Rightarrow \boxed{\mu = E(X) = np}$$

Variance. 2011, 2012

$$V(X) = E(X^2) - (E(X))^2 \quad \text{——— (1)}$$

Since $X^2 = X(X-1) + X$

Therefore, $E(X^2) = E(X(X-1)) + E(X)$ — (2)

Note that

$$E(X(X-1)) = \sum_{x=0}^{n} x(x-1) p^{x} q^{n-x} \binom{n}{x}$$

$$= \sum_{x=0}^{n} \frac{n!}{x!\,(n-x)!} x(x-1) p^{x} q^{n-x}$$

$$= \sum_{x=0}^{n} \frac{\cancel{x(x-1)}\, n(n-1)(n-2)!}{\cancel{x(x-1)}(x-2)!\,(n-x)!} p^{x-2} p^{2} q^{n-x}$$

$$= np^{2} \sum_{x=2}^{n} (n-1) \frac{(n-2)!}{(x-2)!\,(n-x)!} p^{x-2} q^{n-x}$$

$$= n(n-1) p^{2} \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!\,(n-x)!} p^{x-2} q^{n-x}$$

$$= n(n-1) p^{2} \sum_{x=2}^{n} \binom{n-2}{x-2} p^{x-2} q^{n-x}.$$

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Let $x-2 = y$ and $n-2 = m$

$\Rightarrow x \to 2 \Rightarrow y \to 0$

$x \to n \Rightarrow y \to m$

Then $E(X(X-1)) = n(n-1)p^2 \sum_{y=0}^{m} \binom{m}{y} p^y q^{m-y}$

$= n(n-1)p^2 (1)$

$\Rightarrow \boxed{E(X(X-1)) = n(n-1)p^2}$

(2) $\Rightarrow E(X^2) = n(n-1)p^2 + np$

Thus

(1) $\Rightarrow V(X) = n(n-1)p^2 + np - n^2p^2$

$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$

$= np(1-p)$

$\Rightarrow \boxed{V(X) = npq}$ $\quad \because p+q=1$

Moments About Origin.

$\mu_r' = E(X-0)^r$

$= E(X^r)$

$r=1 \Rightarrow \mu_1' = E(X) = np$

$r=2 \Rightarrow \mu_2' = E(X)^2 = n(n-1)p^2 + np$

$r=3 \Rightarrow \mu_3' = E(X^3)$

Since

$X^3 = X(X-1)(X-2) + 3X(X-1) + X$

Therefore

$E(X^3) = E(X(X-1)(X-2)) + 3E(X(X-1)) + E(X)$

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Note that

$$E(x(x-1)(x-2)) = \sum_{x=0}^{n} x(x-1)(x-2)\binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1)(x-2) \frac{n!}{x!\,(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \frac{\cancel{x(x-1)(x-2)}\, n(n-1)(n-2)(n-3)!}{\cancel{x(x-1)(x-2)}(x-3)!\,(n-x)!} p^{x-3} p^3 q^{n-x}$$

$$= n(n-1)(n-2)p^3 \sum_{x=3}^{n} \frac{(n-3)!}{(x-3)!\,(n-x)!} p^{x-3} q^{n-x}$$

$$= n(n-1)(n-2)p^3 \sum_{x=3}^{n} \binom{n-3}{x-3} p^{x-3} q^{n-x}$$

$$= n(n-1)(n-2)p^3 \sum_{y=0}^{m} \binom{m}{y} p^y q^{m-y}$$ After letting

$$= n(n-1)(n-2)p^3 (1)$$

$$\Rightarrow \boxed{E(x(x-1)(x-2)) = n(n-1)(n-2)p^3}$$

Thus,

$$\mu_3' = E(x^3) = n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$$

$$\Rightarrow \boxed{\mu_3' = n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np}$$

$r=4 \Rightarrow \mu_4' = E(x^4)$.

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Since $X^4 = X(X-1)(X-2)(X-3) + 6X(X-1)(X-2)$
$+ 7X(X-1) + X$

So $E(X^4) = E(X(X-1)(X-2)(X-3)) +$
$6E(X(X-1)(X-2)) + 7E(X(X-1)) + E(X)$.

Note that

$$E(X(X-1)(X-2)(X-3)) = \sum_{x=0}^{n} x(x-1)(x-2)(x-3)\binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1)(x-2)(x-3)\frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= n(n-1)(n-2)(n-3)p^4$$ After calculating

Thus μ_4'

$$\boxed{E(X^4) = n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np}$$

Relationship between μ_r and μ_r'.

$\mu_1 = 0$ (always)

$\mu_2 = \mu_2' - (\mu_1')^2$ (Variance)

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$

Moments About Mean.

A/2018 $\mu_r = E(X-\mu)^r$

A/2015 first 3 momets about mean

where μ is mean.

$\mu_1 = E(X-\mu)$

$= E(X) - \mu = \mu - \mu = 0$

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$$\mu_2 = \mu_2' - (\mu_1')^2$$
$$= n(n-1)p^2 + np - n^2p^2$$
$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$
$$= np(1-p)$$
$$\boxed{\mu_2 = npq} \qquad \because 1-p = q$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$
$$= n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$$
$$-3\{n(n-1)p^2 + np\}(np) + 2(np)^3$$

$$= n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$$
$$-3n^2(n-1)p^3 - 3n^2p^2 + 2n^3p^3$$

$$= np\{(n-1)(n-2)p^2 + 3(n-1)p + 1$$
$$-3n(n-1)p^2 - 3np + 2n^2p^2\}$$

$$= np\{(n^2-3n+2)p^2 + \cancel{3np} - 3p + 1$$
$$-(3n^2-3n)p^2 - \cancel{3np} + 2n^2p^2\}.$$

$$= np\{\cancel{n^2p^2} - \cancel{3np^2} + 2p^2 - 3p + 1$$
$$-\cancel{3n^2p^2} + \cancel{3np^2} + \cancel{2n^2p^2}\}.$$

$$= np(2p^2 - 3p + 1)$$
$$= np(2p^2 - 2p - p + 1)$$
$$= np\{2p(p-1) - 1(p-1)\}$$

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$$= np\{(p-1)(2p-1)\}$$

$$= np\{(1-p)(1-2p)\}$$

$$= npq(1-p-p)$$

$$= npq(q-p) \qquad \because q+p=1$$

$$\Rightarrow \boxed{\mu_3 = npq(q-p)}$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$

$$= n(n-1)(n-2)(n-3)p^4 + 6\{n(n-1)(n-2)p^3\}$$
$$+ 7n(n-1)p^2 + np - 4\{n(n-1)(n-2)p^3 +$$
$$3n(n-1)p^2 + np\}\{np\} + 6\{n(n-1)p^2 + np\}(np)^2$$
$$-3(np)^4$$

$$= np\{(n-1)(n-2)(n-3)p^3 + 6(n-1)(n-2)p^2 +$$
$$7(n-1)p + 1 - 4n(n-1)(n-2)p^3 - 12n(n-1)p^2$$
$$-4np + 6n^2(n-1)p^3 + 6n^2p^2 - 3n^3p^3\}.$$

$$= np\{(n^2-3n+2)(n-3)p^3 + 6(n^2-3n+2)p^2$$
$$+ (7n-7)p + 1 - 4n(n^2-3n+2)p^3 - 12(n^2-n)p^2$$
$$-4np + 6(n^3-n^2)p^3 + 6n^2p^2 - 3n^3p^3\}$$

$$= np\{\cancel{n^3p^3} - \cancel{3n^2p^3} + 2np^3 - \cancel{3n^2p^3} + 9np^3 - 6p^3$$
$$+ \cancel{6n^2p^2} - 18np^2 + 12p^2 + 7np - 7p + 1$$
$$-\cancel{4n^3p^3} + \cancel{12n^2p^3} - 8np^3 - \cancel{12n^2p^2} + 12np^2$$
$$-4np + \cancel{6n^3p^3} - \cancel{6n^2p^3} + \cancel{6n^2p^2} - \cancel{3n^3p^3}\}$$

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$$= np\{3np^3 - 6p^3 - 6np^2 + 12p^2 + 3np - 7p + 1\}.$$

$$= np\{3np(p^2 - 2p + 1) - 6p^3 + 12p^2 - 7p + 1\}$$

$$= np\{3np(1-p)^2 + 6p^2(-p + 2) - 6p - p + 1\}$$

$$= np\{3npq^2 + 6p^2(1 - p + 1) - 6p\ 1 - p\}.$$

$$= np\{3npq^2 + 6p^2(q + 1) - 6p + q\}.$$

$$= np\{3npq^2 + 6p^2q + 6p^2 - 6p + q\}.$$

$$= np\{3npq^2 + 6p^2q + 6p(p - 1) + q\}$$

$$= np\{3npq^2 + 6p^2q - 6pq + q\} \quad \because p - 1 = -q$$

$$= npq\{3npq + 6p^2 - 6p + 1\}.$$

$$= npq\{3npq + 6p(p - 1) + 1\}.$$

$$= npq\{3npq - 6pq + 1\}$$

$$= npq\{1 + (3n - 6)pq\}$$

$$= npq\{1 + 3(n - 2)pq\}.$$

$$\Rightarrow \boxed{\mu_4 = npq\{1 + 3(n-2)pq\}}$$

M.G.F. About ~~Mean~~ Origin.

$$M_0(t) = E(e^{tx})$$

$$= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x q^{n-x}, \quad x = 0, 1, \cdots n.$$

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$$= \sum_{x=0}^{n} \frac{n!}{x!\,(n-x)!} (pe^t)^x q^{n-x}$$

$$= n! \sum_{x=0}^{n} \frac{1}{x!\,(n-x)!} (pe^t)^x q^{n-x}$$

$$= n! \left\{ \frac{1}{n!} (pe^t)^0 q^n + \frac{1}{1!(n-1)!} pe^t q^{n-1} + \frac{1}{2!\,(n-2)!} (pe^t)^2 q^{n-2} + \dots + \frac{1}{n!} (pe^t)^n q^0 \right\}$$

$$= q^n (pe^t)^0 + n q^{n-1} (pe^t)^1 + \frac{n(n-1)}{2!} q^{n-2} (pe^t)^2 + \dots + \frac{n!}{n!} q^0 (pe^t)^n$$

$$= (q + pe^t)^n$$ using binomial theorem.

$$\Rightarrow \boxed{M_0(t) = (q + pe^t)^n}$$

$$\mu_r' = \frac{d^r}{dt^r} (q + pe^t)^n \Big|_{t=0}$$

$$\Rightarrow \mu_1' = \frac{d}{dt} (q + pe^t)^n \Big|_{t=0}$$

$$= n (q + pe^t)^{n-1} (pe^t) \Big|_{t=0}$$

$$= n (q + p)^{n-1} p$$

$$= n (1)^{n-1} p$$

$$= np$$

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$$\mu_2' = \frac{d^2}{dt^2}(q+pe^t)^n\Big|_{t=0}$$

$$= \frac{d}{dt} np\, e^t (q+pe^t)^{n-1}\Big|_{t=0}$$

$$= np\, e^t (q+pe^t)^{n-1}\Big|_{t=0} + np\, e^t (n-1)(q+pe^t)^{n-2}(pe^t)\Big|_{t=0}$$

$$= np(q+p)^{n-1} + n(n-1)p^2(q+p)^{n-2}$$

$$= np(1) + n(n-1)p^2(1)$$

$$= n(n-1)p^2 + np$$

$$\mu_3' = \frac{d^3}{dt^3}(q+pe^t)^n\Big|_{t=0}$$

$$= \frac{d}{dt}\left\{ np\, e^t (q+pe^t)^{n-1} + np^2(n-1)e^{2t}(q+pe^t)^{n-2} \right\}\Big|_{t=0}$$

$$= \left\{ \frac{d}{dt} np\, e^t (q+pe^t)^{n-1} + \frac{d}{dt} n(n-1)p^2 e^{2t}(q+pe^t)^{n-2} \right\}\Big|_{t=0}$$

$$= \Big\{ npe^t(n-1)(q+pe^t)^{n-2}(pe^t) + npe^t(q+pe^t)^{n-1}$$

$$+ n(n-1)p^2 e^{2t}(n-2)(q+pe^t)^{n-3}(pe^t)$$

$$+ n(n-1)p^2\, 2e^{2t}(q+pe^t)^{n-2} \Big\}\Big|_{t=0}$$

$$= \Big\{ n(n-1)p^2 e^{2t}(q+pe^t)^{n-2} + npe^t(q+pe^t)^{n-1}$$

$$+ n(n-1)(n-2)p^3 e^{3t}(q+pe^t)^{n-3} + 2n(n-1)p^2 e^{2t}(q+pe^t)^{n-2} \Big\}\Big|_{t=0}$$

$$= \Big\{ 3n(n-1)p^2 e^{2t}(q+pe^t)^{n-2} + npe^t(q+pe^t)^{n-1}$$

$$+ n(n-1)(n-2)p^3 e^{3t}(q+pe^t)^{n-3} \Big\}\Big|_{t=0}$$

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$$= 3n(n-1)p^2(q+p)^{n-2} + np(q+p)^{n-1}$$
$$+ n(n-1)(n-2)p^3(p+q)^{n-3}$$

$$= 3n(n-1)p^2 + np + n(n-1)(n-2)p^3$$

$$\Rightarrow \mu_3' = n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np.$$

$$\mu_4' = \frac{d^4}{dt^4}(q+pe^t)^n\Big|_{t=0}$$

$$= \frac{d}{dt}\Big\{3n(n-1)p^2e^{2t}(q+pe^t)^{n-2} + npe^t(q+pe^t)^{n-1}$$
$$+ n(n-1)(n-2)p^3e^{3t}(q+pe^t)^{n-3}\Big\}\Big|_{t=0}$$

$$= \Big\{3n(n-1)p^2e^{2t}(n-2)(q+pe^t)^{n-3}pe^t + 3n(n-1)p^2 2e^{2t}$$
$$(q+pe^t)^{n-2} + npe^t(n-1)(q+pe^t)^{n-2}pe^t$$
$$+ npe^t(q+pe^t)^{n-1} + n(n-1)(n-2)p^3 3e^{3t}(q+pe^t)^{n-3}$$
$$+ n(n-1)(n-2)(n-3)p^3e^{3t}(q+pe^t)^{n-4}pe^t\Big\}\Big|_{t=0}$$

$$= \Big\{3n(n-1)p^3e^{3t}(n-2)(q+pe^t)^{n-3} + 6n(n-1)p^2e^{2t}(q+pe^t)^{n}$$
$$+ n(n-1)p^2e^{2t}(q+pe^t)^{n-2} + npe^t(q+pe^t)^{n-1}$$
$$+ 3n(n-1)(n-2)p^3e^{3t}(q+pe^t)^{n-3} +$$
$$n(n-1)(n-2)(n-3)p^4e^{4t}(q+pe^t)^{n-4}\Big\}\Big|_{t=0}$$

$$= n(n-1)(n-2)(n-3)p^4 + 3n(n-1)(n-2)p^3 + np$$
$$n(n-1)p^2 + 6n(n-1)p^2 + 3n(n-1)(n-2)p^3$$

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$$\Rightarrow \mu_4' = n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np$$

M.G.F About Mean.

A/2011 also find 1st 4 momets about mean.

$$M_\mu(t) = E(e^{t(x-\mu)})$$

$$= \sum_{x=0}^{n} e^{t(x-\mu)} \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} e^{tx} \cdot e^{-\mu t} \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} e^{-\mu t} e^{tx} \binom{n}{x} p^x q^{n-x}$$

$$= e^{-\mu t} \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x q^{n-x}$$

$$= e^{-\mu t} M_0(t)$$

$$= e^{-\mu t} (q + pe^t)^n$$

$$= e^{-npt} (q + pe^t)^n \qquad \because \mu = np$$

$$= [e^{-pt} (q + pe^t)]^n$$

$$= (q e^{-pt} + p e^{-pt} e^t)^n$$

$$= [(q e^{-pt} + p e^{t(1-p)}]^n$$

$$= (q e^{-pt} + p e^{qt})^n \qquad \because 1-p = q$$

$$\boxed{M_\mu(t) = (q e^{-pt} + p e^{qt})^n}$$

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$$\mu_1 = \frac{d^1}{dt^1} M_x(t)\Big|_{t=0}$$

$$= \frac{d^1}{dt^1}\left(qe^{-pt} + pe^{qt}\right)^n\Big|_{t=0}$$

$$r=1 \Rightarrow \mu_1 = \frac{d}{dt}\left(qe^{-pt} + pe^{qt}\right)^n\Big|_{t=0}$$

$$= n\left(qe^{-pt} + pe^{qt}\right)^{n-1}\left(-pq\,e^{-pt} + pq\,e^{qt}\right)\Big|_{t=0}$$

$$= n(q+p)^{n-1}(-pq + pq) = 0$$

$$\Rightarrow \mu_1 = 0$$

$$\mu_2 = \frac{d^2}{dt^2}\left(qe^{-pt} + pe^{qt}\right)^n\Big|_{t=0}$$

$$= \frac{d}{dt}\, npq\left(qe^{-pt} + pe^{qt}\right)^{n-1}\left(-e^{-pt} + e^{qt}\right)\Big|_{t=0}$$

$$= npq(n-1)\left(qe^{-pt} + pe^{qt}\right)^{n-2}\left(-pqe^{-pt} + pqe^{qt}\right)\cdot\left(-e^{-pt} + e^{qt}\right) + npq\left(qe^{-pt} + pe^{qt}\right)^{n-1}\left(+pe^{-pt} + qe^{qt}\right)\Big|_{t=0}$$

$$= np^2q^2(n-1)\left(qe^{-pt} + pe^{qt}\right)^{n-2}\left(-e^{-pt} + e^{qt}\right)^2\Big|_{t=0}$$

$$+ npq\left(qe^{-pt} + pe^{qt}\right)^{n-1}\left(pe^{-pt} + qe^{qt}\right)\Big|_{t=0}$$

$$= np^2q^2(n-1)(q+p)^{n-2}(-1+1)^2 +$$

$$npq(q+p)^{n-1}(p+q)$$

$$\Rightarrow \mu_2 = npq(1) = npq = Var(X)$$

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$$\mu_3 = \frac{d^3}{dt^3}\left(qe^{-pt}+pe^{qt}\right)^n\Big|_{t=0}$$

$$= \frac{d}{dt}\Big[np^2q^2(n-1)\left(qe^{-pt}+pe^{qt}\right)^{n-2}\left(e^{qt}-e^{-pt}\right)^2$$
$$+npq\left(qe^{-pt}+pe^{qt}\right)^{n-1}\left(pe^{-pt}+qe^{qt}\right)\Big]\Big|_{t=0}$$

$$=\Big\{np^2q^2(n-1)\left(qe^{-pt}+pe^{qt}\right)^{n-2}2\left(e^{qt}-e^{-pt}\right)\left(qe^{qt}+pe^{-pt}\right)$$
$$+np^2q^2(n-1)(n-2)\left(qe^{-pt}+pe^{qt}\right)^{n-3}\left(-pqe^{-pt}+pqe^{qt}\right)$$
$$\left(e^{qt}-e^{-pt}\right)^2+npq\left(qe^{-pt}+pe^{qt}\right)^{n-1}\left(-p^2e^{-pt}+q^2e^{qt}\right)$$
$$+npq(n-1)\left(qe^{-pt}+pe^{qt}\right)^{n-2}\left(-pqe^{-pt}+pqe^{qt}\right)$$
$$\left(pe^{-pt}+qe^{qt}\right)\Big\}\Big|_{t=0}$$

$$=\Big[2np^2q^2(n-1)\left(qe^{-pt}+pe^{qt}\right)^{n-2}\left(e^{qt}-e^{-pt}\right)\left(qe^{qt}+pe^{-pt}\right)$$
$$+np^3q^3(n-1)(n-2)\left(qe^{-pt}+pe^{qt}\right)^{n-3}\left(e^{qt}-e^{-pt}\right)^3$$
$$+npq\left(qe^{-pt}+pe^{qt}\right)^{n-1}\left(-p^2e^{-pt}+q^2e^{qt}\right)$$
$$+np^2q^2(n-1)\left(qe^{-pt}+pe^{qt}\right)^{n-2}\left(e^{qt}-e^{-pt}\right)\times$$
$$\left(pe^{-pt}+qe^{qt}\right)\Big]\Big|_{t=0}$$

$$=2np^2q^2(n-1)(q+p)^{n-2}(1-1)(q+p)$$
$$+np^3q^3(n-1)(n-2)(q+p)^{n-3}(1-1)^3$$
$$+npq(q+p)^{n-1}(-p^2+q^2)+n(n-1)p^2q^2(q+p)^{n-2}$$
$$(1-1)(p+q)$$

$$=npq(1)(q^2-p^2) = npq(q-p)(q+p)$$
$$=npq(q-p)$$

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$$\mu_4 = \frac{d^4}{dt^4}\left(qe^{-pt}+pe^{qt}\right)^n\Big|_{t=0}$$

$$= \frac{d}{dt}\Big[2n(n-1)p^2q^2\left(qe^{-pt}+pe^{qt}\right)^{n-2}\left(e^{qt}-e^{-pt}\right)$$
$$\left(qe^{qt}+pe^{-pt}\right)+n(n-1)(n-2)p^3q^3\left(qe^{-pt}+pe^{qt}\right)^{n-3}$$
$$\left(e^{qt}-e^{-pt}\right)^3+npq\left(qe^{-pt}+pe^{qt}\right)^{n-1}\left(-pe^{-pt}+q^2e^{qt}\right)$$
$$+np^2q^2(n-1)\left(qe^{-pt}+pe^{qt}\right)^{n-2}\left(e^{qt}-e^{-pt}\right)\times$$
$$\left(pe^{-pt}+qe^{qt}\right)\Big]_{t=0}$$

$$=\Big[2n(n-1)p^2q^2\left(qe^{-pt}+pe^{qt}\right)^{n-2}\left(e^{qt}-e^{-pt}\right)$$
$$\left(q^2e^{qt}-p^2e^{-pt}\right)+2n(n-1)p^2q^2\left(qe^{-pt}+pe^{qt}\right)^{n-2}$$
$$\left(qe^{qt}+pe^{-pt}\right)\left(qe^{qt}+pe^{-pt}\right)+2n(n-1)p^2q^2$$
$$(n-2)\left(qe^{-pt}+pe^{qt}\right)^{n-3}\left(-pqe^{-pt}+pqe^{qt}\right)\left(e^{qt}-e^{-pt}\right)$$
$$+n(n-1)(n-2)p^3q^3\left(qe^{-pt}+pe^{qt}\right)^{n-3}3\left(e^{qt}-e^{-pt}\right)^2\left(qe^{qt}+pe^{-pt}\right)$$
$$+n(n-1)(n-2)p^3q^3(n-3)\left(qe^{-pt}+pe^{qt}\right)^{n-4}\left(-pqe^{-pt}+pqe^{qt}\right)$$
$$\left(e^{qt}-e^{-pt}\right)+npq\left(qe^{-pt}+pe^{qt}\right)^{n-1}\left(+p^3e^{-pt}+q^3e^{qt}\right)$$
$$+npq(n-1)\left(qe^{-pt}+pe^{qt}\right)^{n-2}\left(-pqe^{-pt}+pqe^{qt}\right)\left(-p^2e^{-pt}+q^2e^{qt}\right)$$
$$+np^2q^2(n-1)\left(qe^{-pt}+pe^{qt}\right)^{n-2}\left(e^{qt}-e^{-pt}\right)\left(-p^2e^{-pt}+q^2e^{qt}\right)$$
$$+n(n-1)p^2q^2\left(qe^{-pt}+pe^{qt}\right)^{n-2}\left(qe^{qt}+pe^{-pt}\right)$$
$$\left(pe^{-pt}+qe^{qt}\right)+n(n-1)p^2q^2(n-2)\left(qe^{-pt}+pe^{qt}\right)^{n-3}$$
$$\left(-pqe^{-pt}+pqe^{qt}\right)\left(e^{qt}+pe^{-pt}\right)\left(pe^{-pt}+qe^{qt}\right)\Big]_{t=0}$$

$$=3n(n-1)p^2q^2(q+p)^{n-2}(1-1)(q^2-p^2)$$
$$+3n(n-1)p^2q^2(q+p)^{n-2}(q+p)(q+p)$$

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$$+3n(n-1)(n-2)p^2q^2(q+p)(-pq+pq)(1-1)$$
$$-3n(n-1)(n-2)p^3q^3(q+p)^{n-3}(1-1)(q+p)$$
$$+n(n-1)(n-2)(n-3)p^3q^3(q+p)^{n-4}(-pq+pq)(1-1)$$
$$+npq(q+p)^{n-1}(p^3+q^3)+n(n-1)pq(q+p)^{n-2}$$
$$(-pq+pq)(-p^2+q^2)$$

$$= 3n(n-1)p^2q^2 + npq(p^3+q^3)$$
$$= npq[p^3+q^3+3(n-1)pq]$$
$$= npq[p^3+(1-p)^3+3npq-3pq]$$
$$= npq[\cancel{p^3}+1-\cancel{p^3}-3p+3p^2+3npq-3pq]$$
$$= npq[1-3p+3p^2+3npq-3pq]$$
$$= npq[1-3p(1-p)+3npq-3pq]$$
$$= npq[1-3pq+3npq-3pq]$$
$$= npq[1+3npq-6pq]$$
$$= npq[1+3(n-2)pq]$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{n^2p^2q^2(q-p)^2}{n^3p^3q^3} = \frac{(q-p)^2}{npq}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{npq(1+3(n-2)pq)}{n^2p^2q^2} = \frac{1+3(n-2)pq}{npq}$$

Recurrence Formula.

We know that

$$P(X=x) = \binom{n}{x}p^x q^{n-x} \quad ; x=0,1,\cdots n$$

$$= \frac{n!}{x!\,(n-x)!}p^x q^{n-x} \qquad (1)$$

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and $P(X=x-1) = \binom{n}{x-1} p^{x-1} q^{n-(x-1)}$

$$= \frac{n!}{(x-1)!\,(n-(x-1))!} p^{x-1} q^{n-(x-1)} \quad \text{—— (2)}$$

Dividing (1) by (2), we get

$$\cancel{\frac{P(X=x)}{P(X=x-1)} = \frac{n!\, p^{x-1} q^{n-(x-1)}}{(x-1)!\,(n-(x-1))!} \times \frac{x!\,(n-x)!}{n!\, p^{x} q^{n-x}}}$$

$$\cancel{= p^{x} p^{-1} q^{n-x} q^{+1} \quad x(x-1)!\,(n-x)(n-x}$$

$$\frac{P(X=x)}{P(X=x-1)} = \frac{n!\, p^{x} q^{n-x}}{x!\,(n-x)!} \times \frac{(x-1)!\,(n-x+1)!}{n!\, p^{x-1} q^{n-x+1}}$$

$$= \frac{\cancel{p^{x}}\, \cancel{q^{n-x}}\, \cancel{(x-1)!}\,(n-x+1)\,\cancel{(n-x)!}}{x\,\cancel{(x-1)!}\,\cancel{(n-x)!}\,\cancel{p^{x}} p^{-1}\, \cancel{q^{n-x}}\, q}$$

$$= \frac{n-x+1}{x\, p^{-1} q}$$

$$= \frac{n-x+1}{x} \, p/q$$

$$\Rightarrow P(X=x) = \frac{n-x+1}{x} \, p/q \; P(X=x-1)$$

Theorem. 2010, 2017 A/2018

For the binomial distribution $(q+p)^n$ prove the relation

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$$\mu_{r+1} = pq\left(nr\,\mu_{r-1} + \frac{d\mu_r}{dp}\right)$$

where μ_r is the rth moment about the mean. Hence find μ_2, μ_3 and μ_4.

Proof.

We know that the rth moment about the mean is

$$\mu_r = E(x-\mu)^r$$

$$= \sum_{x=0}^{n} (x-\mu)^r \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} (x-np)^r \binom{n}{x} p^x (1-p)^{n-x} \qquad \because q = 1-p,\ \mu = np$$

Diff. w.r.t. p, we get

$$\frac{d\mu_r}{dp} = \sum_{x=0}^{n} (x-np)^r \binom{n}{x} \left\{ p^x (n-x)(1-p)^{n-x-1}(-1) + x p^{x-1}(1-p)^{n-x} \right\} + \sum_{x=0}^{n} r(x-np)^{r-1}(-n)\binom{n}{x} p^x (1-p)^{n-x}$$

$$= -\sum_{x=0}^{n} (n-x)(x-np)^r \binom{n}{x} p^x q^{n-x-1} +$$

$$\sum_{x=0}^{n} x(x-np)^r \binom{n}{x} p^{x-1} q^{n-x}$$

$$-nr\sum_{x=0}^{n} (x-np)^{r-1} \binom{n}{x} p^x q^{n-x}$$

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$$= \sum_{x=0}^{n} (x-np)^{s} \binom{n}{x} p^{x} q^{n-x} \left\{ -(n-x) q^{-1} + x p^{-1} \right\}$$

$$- n s \mu_{s-1}$$

$$= \sum_{x=0}^{n} (x-np)^{s} \binom{n}{x} p^{x} q^{n-x} \left(\frac{x-n}{q} + \frac{x}{p} \right)$$

$$- n s \mu_{s-1}$$

$$= \sum_{x=0}^{n} (x-np)^{s} \binom{n}{x} p^{x} q^{n-x} \left(\frac{px - np + xq}{pq} \right)$$

$$- n s \mu_{s-1}$$

$$\Rightarrow \frac{d\mu_s}{dp} + n s \mu_{s-1} = \sum_{x=0}^{n} (x-np)^{s} \binom{n}{x} p^{x} q^{n-x} \left(\frac{x-np}{pq} \right)$$

$$\Rightarrow \frac{d\mu_s}{dp} + n s \mu_{s-1} = \frac{1}{pq} \sum_{x=0}^{n} (x-np)^{s+1} \binom{n}{x} p^{x} q^{n-x}$$

$$\Rightarrow \frac{d\mu_s}{dp} + n s \mu_{s-1} = \frac{1}{pq} \mu_{s+1}$$

Hence,

$$\mu_{s+1} = pq \left(n s \mu_{s-1} + \frac{d\mu_s}{dp} \right)$$

Now, we find μ_2, μ_3 and μ_4 as follows:

Since $\mu_1 = 0$ and $\mu_0 = E(X^0) = 1$

So $\mu_2 = pq \left(n(1) \mu_0 + \frac{d\mu_1}{dp} \right)$

$$= pq (n + 0)$$

$$= npq$$

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$$\mu_3 = pq\left(2n\mu_1 + \frac{d\mu_2}{dp}\right)$$

$$= pq\left(2n(0) + \frac{d\,npq}{dp}\right)$$

$$= pq\,(n)\frac{d}{dp}p(1-p) \qquad \because q = 1-p$$

$$= npq\frac{d}{dp}(p-p^2)$$

$$= npq(1-2p) \quad = npq(1-p-p)$$

$$= npq(q-p).$$

$$\mu_4 = pq\left(3n\mu_2 + \frac{d\mu_3}{dp}\right)$$

$$= pq\left(3n(npq) + \frac{d}{dp}npq(q-p)\right)$$

$$= pq\left\{3n^2pq + n\frac{d}{dp}p(1-p)(1-2p)\right\}.$$

$$= npq\left\{3npq + (1-p)(1-2p) + p(-1)(1-2p) + p(1-p)(-2)\right\}$$

$$= npq\left\{3npq + q(q-p) - p(q-p) - 2p(q)\right\}$$

$$= npq\left\{3npq + q^2 - qp - pq + p^2 - 2pq\right\}$$

$$= npq\left\{3npq + (1-p)^2 + p^2 - 4pq\right\}$$

$$= npq\left\{3npq + 1 + p^2 - 2p + p^2 - 4pq\right\}$$

$$= npq\left\{3npq + 1 + 2p^2 - 2p - 4pq\right\}$$

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$= npq\{3npq + 1 + 2p(p-1) - 4pq\}$.

$= npq\{3npq + 1 - 2pq - 4pq\}$.

$= npq\{3npq + 1 - 6pq\}$.

$= npq\{1 + 3(n-2)pq\}$.

Cumulant Generating Function.

2013 The cumulant G.F. is given by

$$K(t) = \ln M_0(t)$$

where

$$M_0(t) = (q + pe^t)^n$$

$$K(t) = \ln (q + pe^t)^n$$

$$= n \ln (q + pe^t)$$

$$= n \ln (1 - p + pe^t)$$

$$= n \ln [1 - p(1 - e^t)]$$

$$= n \ln \left[1 - p\left(1 - \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)\right)\right]$$

$$= n \ln \left[1 + p\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)\right]$$

$$\Rightarrow K(t) = n \ln \left[1 + \left\{pt + \frac{pt^2}{2!} + \frac{pt^3}{3!} + \cdots\right\}\right]$$

Since we know that

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Therefor,

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$$K(t) = n\left\{p\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots\right) - \frac{p^2}{2}\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots\right)^2\right.$$

$$\left. + \frac{p^3}{3}\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots\right)^3 - \frac{p^4}{4}\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots\right)^4+\cdots\right\}$$

$$= n\left\{p\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots\right) - \frac{p^2}{2}\left(t^2+\frac{t^4}{4}+\frac{2t^3}{2!}+\frac{2t^4}{3!}\right.\right.$$

$$\left.\left. +\cdots\right) + \frac{p^3}{3}\left(t^3+3\frac{t^4}{2!}+\cdots\right) - \frac{p^4}{4}\left(t^4+\cdots\right)+\cdots\right\}$$

$$= n\left\{\frac{pt}{1!} + (p-p^2)\frac{t^2}{2!} + (p-3p^2+2p^3)\frac{t^3}{3!}\right.$$

$$\left. + (p-7p^2+12p^3-6p^4)\frac{t^4}{4!}+\cdots\right\}$$

As we know that, the general expression for cumulant is

$$K(t) = K_1\frac{t}{1!} + K_2\frac{t^2}{2!} + K_3\frac{t^3}{3!} + K_4\frac{t^4}{4!}+\cdots$$

Therefore,

$$K_1\frac{t}{1!} + K_2\frac{t^2}{2!} + K_3\frac{t^3}{3!} + \frac{K_4}{4!}t^4+\cdots = np\frac{t}{1!} + n(p-p^2)\frac{t^2}{2!}$$

$$+ n(p-3p^2+2p^3)\frac{t^3}{3!} + n(p-7p^2+12p^3-6p^4)\frac{t^4}{4!}+\cdots$$

Comparing like powers of t, we have

$K_1 = np$

$K_2 = n(p-p^2) = np(1-p) = npq$

$K_3 = n(p-3p^2+2p^3)$

$= np(1-3p+2p^2) = np(2p^2-2p-p+1)$

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$$= np[2p(p-1) - 1(p-1)]$$
$$= np[(2p-1)(p-1)]$$
$$= np(1-2p)(1-p)$$
$$= npq(q-p)$$

By synthetic division

	-6	12	-7	1
1	↓	-6	6	-1
	-6	6	-1	0

$$K_4 = n(p - 7p^2 + 12p^3 - 6p^4)$$
$$= np(1 - 7p + 12p^2 - 6p^3)$$
$$= np(p-1)(-6p^2 + 6p - 1)$$

$$= np(1-p)(6p^2 - 6p + 1)$$
$$= npq(6p(p-1) + 1)$$
$$= npq(1 - 6pq) \quad \because p-1 = -q$$

Question.

Show that for the binomial dis. $(q+p)^n$,

$$K_{r+1} = pq\frac{dK_r}{dp} \quad ; r \geq 1$$

and hence find out the 1st four cumulants.

Sol.

We know that the rth cumulant gen. function is

$$K_r = \frac{d^r}{dt^r} \ln M_0(t)\Big|_{t=0}$$
$$= \frac{d^r}{dt^r} \ln(q + pe^t)^n\Big|_{t=0}$$

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$$= n \frac{d^r}{dt^r} \ln(q+pe^t)\Big|_{t=0}$$

Diff. w.r.t. p, we get

$$\frac{dK_r}{dp} = n \frac{d^r}{dt^r} \frac{d}{dp} \ln(q+pe^t)\Big|_{t=0}$$

$$= n \frac{d^r}{dt^r} \frac{d}{dp} \ln(1-p+pe^t)\Big|_{t=0}$$

$$= n \frac{d^r}{dt^r} \frac{1}{1-p+pe^t} \cdot (-1+e^t)\Big|_{t=0}$$

$$\Rightarrow \frac{dK_r}{dp} = n \frac{d^r}{dt^r} \frac{(-1+e^t)}{q+pe^t}\Big|_{t=0}$$

$$\Rightarrow pq \frac{dK_r}{dp} = np \frac{d^r}{dt^r} \frac{(-q+qe^t)}{q+pe^t}\Big|_{t=0} \quad \text{——— (1)}$$

Also, we know that

$$K_{r+1} = \frac{d^{r+1}}{dt^{r+1}} \ln M_0(t)\Big|_{t=0}$$

$$= \frac{d^{r+1}}{dt^{r+1}} \ln(q+pe^t)^n\Big|_{t=0}$$

$$= n \frac{d^r}{dt^r} \frac{d}{dt} \ln(q+pe^t)\Big|_{t=0}$$

$$= n \frac{d^r}{dt^r} \frac{1}{q+pe^t} \cdot pe^t\Big|_{t=0}$$

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$$\Rightarrow K_{r+1} = np \frac{d^r}{dt^r} \frac{e^t}{q+pe^t}\Big|_{t=0} \quad \text{——— (2)}$$

Subtracting (1) from (2), we get

$$K_{r+1} - pq\frac{dK_r}{dp} = np\frac{d^r}{dt^r}\frac{e^t}{q+pe^t}\Big|_{t=0} - np\frac{d^r}{dt^r}\left(\frac{-q+qe^t}{q+pe^t}\right)\Big|_{t=0}$$

$$= np\frac{d^r}{dt^r}\left(\frac{e^t+q-qe^t}{q+pe^t}\right)\Big|_{t=0}$$

$$= np\frac{d^r}{dt^r}\frac{(1-q)e^t+q}{q+pe^t}\Big|_{t=0}$$

$$= np\frac{d^r}{dt^r}\frac{\cancel{q+pe^t}}{\cancel{q+pe^t}}\Big|_{t=0} \qquad \because 1-q=p$$

$$= np\frac{d^r}{dt^r}(1)\Big|_{t=0}$$

$$= np(0) = 0$$

$$\Rightarrow K_{r+1} - pq\frac{dK_r}{dp} = 0$$

$$\Rightarrow K_{r+1} = pq\frac{dK_r}{dp}$$

Now we find 1st four cumulants as follow:

$$r = 1 \Rightarrow K_2 = pq\frac{dK_1}{dp}$$

Since $K_1 = np$

therefore, $K_2 = pq\frac{dK_1}{dp} = pq\frac{d(np)}{dp} = npq$

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$$K_3 = pq\frac{dK_2}{dp} = pq\frac{d(npq)}{dp}$$

$$= pq\frac{d}{dp}np(1-p) = npq\frac{d}{dp}(p-p^2)$$

$$= npq(1-2p) = npq(q-p)$$

$$K_4 = pq\frac{dK_3}{dp} = pq\frac{d(npq)(q-p)}{dp}$$

$$= npq\frac{d}{dp}p(1-p)(1-2p)$$

$$= npq\{p(1-p)(-2) + p(-1)(1-2p) + (1-p)(1-2p)\}$$

$$= npq\{pq(-2) - p(q-p) + q(q-p)\}$$

$$= npq\{-2pq + (q-p)(q-p)\}$$

$$= npq\{(q-p)^2 - 2pq\}$$

$$= npq\{q^2 + p^2 - 2pq - 2pq\}$$

$$= npq\{(1-p)^2 + p^2 - 4pq\}$$

$$= npq\{1 + p^2 - 2p + p^2 - 4pq\}$$

$$= npq\{1 + 2p^2 - 2p - 4pq\}$$

$$= npq\{1 + 2p(p-1) - 4pq\}$$

$$= npq\{1 - 2pq - 4pq\}$$

$$= npq\{1 - 6pq\}$$ As required.

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Poisson Distribution.

It is a limiting case of binomial dis. when n is very large and p is very small.

Formula.

$$P(X=x) = \frac{\mu^x}{x!} e^{-\mu} \quad ; x = 0, 1, 2 \cdots$$

where μ is the parameter.

This dis. is denoted by $p(x; \mu)$.

Show that Poisson distribution is limiting case of binomial distribution.

For binomial distribution, we know that

$$b(x; n, p) = \binom{n}{x} p^x q^{n-x} \quad ; x = 0, 1, 2 \cdots n$$

$$= \frac{n!}{x!\,(n-x)!} p^x q^{n-x}$$

$$= \frac{n(n-1)(n-2)\cdots(n-(x-1))\cancel{(n-x)!}}{x!\,\cancel{(n-x)!}} p^x q^{n-x}$$

$$= \frac{n(n-1)(n-2)\cdots(n-(x-1))}{x!} p^x q^{n-x}$$

We know that in binomial distribution

$$\mu = np \Rightarrow p = \mu/n$$

and $q = 1-p \Rightarrow q = 1 - \mu/n$

so

$$b(x; n, p) = \frac{n(n-1)(n-2)\cdots(n-(x-1))}{x!} \left(\frac{\mu}{n}\right)^x \left(1 - \mu/n\right)^{n-x}$$

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$$= \frac{\mu^x}{n^x} \frac{n(n-1)(n-2)\cdots(n-(x-1))}{x!} \left(1-\frac{\mu}{n}\right)^n \left(1-\frac{\mu}{n}\right)^{-x}$$

$$= \frac{\mu^x}{x!} \frac{n(n-1)(n-2)\cdots(n-(x-1))}{n^x} \left(1-\frac{\mu}{n}\right)^n \left(1-\frac{\mu}{n}\right)^{-x}$$

$$= \frac{\mu^x}{x!} 1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)\left(1-\frac{\mu}{n}\right)^n \left(1-\frac{\mu}{n}\right)^{-x}$$

as $n^x = n.n.\cdots n$ $\because x$ - times.

Taking $\lim\limits_{n\to\infty}$, we have

$$\lim_{n\to\infty} b(x;n,p) = \frac{\mu^x}{x!} \lim_{n\to\infty} 1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots$$

$$\left(1-\frac{x-1}{n}\right)\left(1-\frac{\mu}{n}\right)^n \left(1-\mu/n\right)^{-x}$$

$$= \frac{\mu^x}{x!} \lim_{n\to\infty} \left(1-\frac{\mu}{n}\right)^n$$

$$= \frac{\mu^x}{x!} e^{-\mu} \qquad \because \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$$

Hence,

$$\lim_{n\to\infty} b(x;n,p) = \frac{\mu^x}{x!} e^{-\mu} = p(x;\mu)$$

Mean. 2010

$$E(x) = \sum_{x=0}^{\infty} x\, p(x;\mu)$$

$$= \sum_{x=0}^{\infty} x \frac{\mu^x}{x!} e^{-\mu}$$

$$= \sum_{x=0}^{\infty} \cancel{x} \frac{\mu^x}{\cancel{x}(x-1)!} e^{-\mu}$$

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$$= \sum_{x=1}^{\infty} \frac{\mu^x}{(x-1)!} e^{-\mu}$$

$$= \mu \sum_{x=1}^{\infty} \frac{\mu^{x-1}}{(x-1)!} e^{-\mu}$$

Put $x-1 = y$ then $x = 1+y$

when $x=1 \Rightarrow y=0$

$x \to \infty \Rightarrow y \to \infty$

$$E(x) = \mu \sum_{y=0}^{\infty} \frac{\mu^y}{y!} e^{-\mu}$$

$$= \mu(1) = \mu$$

$$\Rightarrow \boxed{E(x) = \mu}$$

Variance. 2010

$$Var(X) = E(X^2) - (E(X))^2$$

Since $X^2 = X(X-1) + X$

so $E(X^2) = E(X(X-1)) + E(X)$.

Note that

$$E(X(X-1)) = \sum_{x=0}^{\infty} x(x-1) \frac{\mu^x}{x!} e^{-\mu}$$

$$= \sum_{x=0}^{\infty} \frac{\cancel{x(x-1)}\, \mu^{x-2} \mu^2}{\cancel{x(x-1)}(x-2)!} e^{-\mu}$$

$$= \mu^2 \sum_{x=2}^{\infty} \frac{\mu^{x-2}}{(x-2)!} e^{-\mu}$$

Put $x-2 = y$ then

$$E(X(X-1)) = \mu^2 \sum_{y=0}^{\infty} \frac{\mu^y}{y!} e^{-\mu}$$

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$= \mu^2$

$\Rightarrow \boxed{E(X(X-1)) = \mu^2}$

$\Rightarrow E(X^2) = \mu^2 + \mu$

Thus, $Var(X) = \cancel{\mu^2} + \mu - \cancel{\mu^2}$

$\Rightarrow \boxed{Var(X) = \mu}$

Moments About Origin.

$\mu_r' = E(X-0)^r$

$\Rightarrow \mu_r' = E(X^r)$

$r=1 \Rightarrow \mu_1' = E(X) = \mu$

$r=2 \Rightarrow \mu_2' = E(X^2) = \mu^2 + \mu$

$r=3 \Rightarrow \mu_3' = E(X^3)$

$= E(X(X-1)(X-2)) + 3E(X(X-1)) + E(X)$

$\Rightarrow \mu_3' = E(X(X-1)(X-2)) + 3\mu^2 + \mu$ ——— (1)

Note that

$$E(X(X-1)(X-2)) = \sum_{x=0}^{\infty} x(x-1)(x-2) \frac{\mu^x}{x!} e^{-\mu}$$

$$= \sum_{x=0}^{\infty} \frac{\cancel{x(x-1)(x-2)}\, \mu^{x-3} \mu^3 e^{-\mu}}{\cancel{x(x-1)(x-2)}(x-3)!}$$

$$= \sum_{x=3}^{\infty} \frac{\mu^{x-3}}{(x-3)!} e^{-\mu}$$

Put $x - 3 = y$ then

$$E(X(X-1)(X-2)) = \sum_{y=0}^{\infty} \frac{\mu^y}{y!} e^{-\mu} \mu^3$$

$$= \mu^3 (1).$$

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$$\Rightarrow \boxed{E(X(X-1)(X-2)) = \mu^3}$$

① $\Rightarrow E(X^3) = \mu^3 + 3\mu^2 + \mu$

$\Rightarrow \mu_3' = \mu^3 + 3\mu^2 + \mu$

and

$$\mu_4' = E(X^4)$$
$$= E(X(X-1)(X-2)(X-3)) + 6E(X(X-1)(X-2))$$
$$+ 7E(X(X-1)) + E(X)$$

We can easily find that

$$E(X(X-1)(X-2)(X-3)) = \mu^4$$

$$\Rightarrow \mu_4' = \mu^4 + 6\mu^3 + 7\mu^2 + \mu$$

Moments About Mean. A/2018

$$\mu_s = E(X-\mu)^s$$

where μ is mean

$$\mu_1 = E(X-\mu)$$
$$= E(X) - \mu = \mu - \mu = 0$$

$$\mu_2 = E(X-\mu)^2$$
$$\mu_2 = \mu_2' - (\mu_1')^2$$
$$= \cancel{\mu^2} + \mu - \cancel{\mu^2} = \mu$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$
$$= \mu^3 + 3\mu^2 + \mu - 3(\mu^2+\mu)(\mu) + 2\mu^3$$
$$= \cancel{\mu^3} + \cancel{3\mu^2} + \mu - \cancel{3\mu^3} - \cancel{3\mu^2} + \cancel{2\mu^3}$$

$$\Rightarrow \mu_3 = \mu$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$

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$$= \mu^4 + 6\mu^3 + 7\mu^2 + \mu - 4(\mu^3 + 3\mu^2 + \mu)(\mu)$$
$$+ 6(\mu^2 + \mu)\mu^2 - 3\mu^4$$
$$= \cancel{\mu^4} + \cancel{6\mu^3} + 7\mu^2 + \mu - \cancel{4\mu^4} - \cancel{12\mu^3} - 4\mu^2$$
$$+ \cancel{6\mu^4} + \cancel{6\mu^3} - \cancel{3\mu^4}$$
$$= 3\mu^2 + \mu$$

M.G.F About Origin. A/2013

$$M_0(t) = E(e^{tx})$$
$$= \sum_{x=0}^{\infty} e^{tx} \frac{\mu^x}{x!} e^{-\mu}$$
$$= \sum_{x=0}^{\infty} \frac{(\mu e^t)^x}{x!} e^{-\mu}$$
$$= e^{-\mu} \sum_{x=0}^{\infty} \frac{(\mu e^t)^x}{x!}$$
$$= e^{-\mu}\left(1 + \frac{\mu e^t}{1!} + \frac{(\mu e^t)^2}{2!} + \cdots\right)$$
$$= e^{-\mu} e^{\mu e^t} = e^{\mu(-1 + e^t)}$$
$$\Rightarrow \boxed{M_0(t) = e^{\mu(e^t - 1)}}$$

$$\mu_r' = \frac{d^r}{dt^r} M_0(t)\Big|_{t=0}$$
$$= \frac{d^r}{dt^r} e^{\mu(e^t - 1)}\Big|_{t=0}$$
$$r=1 \Rightarrow \mu_1' = \frac{d}{dt} e^{\mu(e^t - 1)}\Big|_{t=0}$$
$$= e^{\mu(e^t - 1)} \mu(e^t)\Big|_{t=0}$$

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$$\Rightarrow \mu_1' = \mu e^{\mu(e^t-1)} e^t \Big|_{t=0}$$

$$= \mu(1) = \mu$$

$$\mu_2' = \frac{d^2}{dt^2} e^{\mu(e^t-1)} \Big|_{t=0}$$

$$= \frac{d}{dt}\left[\mu e^{\mu(e^t-1)} e^t\right]_{t=0}$$

$$= \mu e^{\mu(e^t-1)} \mu e^{2t} + \mu e^{\mu(e^t-1)} e^t \Big|_{t=0}$$

$$= \mu e^{\mu(e^t-1)} (\mu e^{2t} + e^t)\Big|_{t=0}$$

$$= \mu(1)(\mu+1) = \mu(\mu+1)$$

$$\Rightarrow \mu_2' = \mu^2 + \mu$$

$$\mu_3' = \frac{d^3}{dt^3} e^{\mu(e^t-1)} \Big|_{t=0}$$

$$= \frac{d}{dt}\left[\mu e^{\mu(e^t-1)} (\mu e^{2t} + e^t)\right]_{t=0}$$

$$= \mu e^{\mu(e^t-1)} (2\mu e^{2t} + e^t) + \mu e^{\mu(e^t-1)} \mu e^t (\mu e^{2t} + e^t)\Big|_{t=0}$$

$$= \mu e^{\mu(e^t-1)} (2\mu e^{2t} + e^t + \mu^2 e^{3t} + \mu e^{2t})\Big|_{t=0}$$

$$= \mu e^{\mu(e^t-1)} (\mu^2 e^{3t} + 3\mu e^{2t} + e^t)\Big|_{t=0}$$

$$= \mu(1)(\mu^2 + 3\mu + 1)$$

$$= \mu^3 + 3\mu^2 + \mu$$

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$$\mu_4' = \frac{d^4}{dt^4} e^{\mu(e^t-1)}\Big|_{t=0}$$

$$= \frac{d}{dt} \mu e^{\mu(e^t-1)} (\mu^2 e^{3t} + 3\mu e^{2t} + e^t)\Big|_{t=0}$$

$$= \Big[\mu e^{\mu(e^t-1)} (3\mu^2 e^{3t} + 6\mu e^{2t} + e^t)$$

$$+ \mu e^{\mu(e^t-1)} \mu e^t (\mu^2 e^{3t} + 3\mu e^{2t} + e^t)\Big]_{t=0}$$

$$= \mu(1)(3\mu^2 + 6\mu + 1) + \mu(1)\mu(\mu^2 + 3\mu + 1)$$

$$= 3\mu^3 + 6\mu^2 + \mu + \mu^2(\mu^2 + 3\mu + 1)$$

$$= 3\mu^3 + 6\mu^2 + \mu + \mu^4 + 3\mu^3 + \mu^2$$

$$= \mu^4 + 6\mu^3 + 7\mu^2 + \mu$$

M.G.F. About Mean. A/2012

$$M_\mu(t) = E(e^{t(x-\mu)})$$

$$= \sum_{x=0}^{\infty} e^{t(x-\mu)} \frac{\mu^x}{x!} e^{-\mu}$$

$$= \sum_{x=0}^{\infty} e^{tx} \cdot e^{-\mu t} \frac{\mu^x}{x!} e^{-\mu}$$

$$= e^{-\mu t} \sum_{x=0}^{\infty} e^{tx} \frac{\mu^x}{x!} e^{-\mu}$$

$$= e^{-\mu t} e^{\mu(e^t-1)}$$

$$= e^{-\mu t + \mu(e^t-1)}$$

$$\boxed{M_\mu(t) = e^{\mu(e^t - t - 1)}}$$

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$$\mu_1 = \frac{d^1}{dt^1} M_\mu(t) \Big|_{t=0}$$

$$= \frac{d^1}{dt^1} e^{\mu(e^t - t - 1)} \Big|_{t=0}$$

$$\mu_1 = \frac{d}{dt} e^{\mu(e^t - t - 1)} \Big|_{t=0}$$

$$= e^{\mu(e^t - t - 1)} \mu(e^t - 1) \Big|_{t=0}$$

$$= (1)\mu(1-1) = 0$$

$$\Rightarrow \mu_1 = 0$$

$$\mu_2 = \frac{d^2}{dt^2} e^{\mu(e^t - t - 1)} \Big|_{t=0}$$

$$= \frac{d}{dt}\left[\mu e^{\mu(e^t - t - 1)} (e^t - 1)\right]_{t=0}$$

$$= \mu e^{\mu(e^t - t - 1)} (e^t) + \mu e^{\mu(e^t - t - 1)} \mu(e^t - 1)^2 \Big|_{t=0}$$

$$= \mu e^{\mu(e^t - t - 1)} (e^t + \mu(e^t - 1)^2) \Big|_{t=0}$$

$$= \mu(1)(1 + \mu(0))$$

$$= \mu$$

$$\Rightarrow \mu_2 = \mu$$

$$\mu_3 = \frac{d^3}{dt^3} e^{\mu(e^t - t - 1)} \Big|_{t=0}$$

$$= \frac{d}{dt}\left[\mu e^{\mu(e^t - t - 1)} (e^t + \mu(e^t - 1)^2)\right]_{t=0}$$

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$$= \frac{d}{dt}\left[\mu e^{\mu(e^t - t - 1)}\left(e^t + \mu(e^{2t} + 1 - 2e^t)\right)\right]_{t=0}$$

$$= \frac{d}{dt}\left[\mu e^{\mu(e^t - t - 1)}\left(e^t + \mu e^{2t} + \mu - 2\mu e^t\right)\right]_{t=0}$$

$$= \frac{d}{dt}\left[\mu e^{\mu(e^t - t - 1)}\left\{\mu e^{2t} + (1-2\mu)e^t + \mu\right\}\right]_{t=0}$$

$$= \mu e^{\mu(e^t - t - 1)}\left\{2\mu e^{2t} + (1-2\mu)e^t + 0\right\}$$

$$+ \mu e^{\mu(e^t - t - 1)} \cdot \left\{\mu(e^t - 1)\right\}\left\{\mu e^{2t} + (1-2\mu)e^t + \mu\right\}\Big|_{t=0}$$

$$= \mu e^{\mu(e^t - t - 1)}\left\{2\mu e^{2t} + (1-2\mu)e^t\right\}$$

$$+ \mu e^{\mu(e^t - t - 1)}\left\{\mu(e^t - 1)\right\}\left\{\mu e^{2t} + (1-2\mu)e^t + \mu\right\}\Big|_{t=0}$$

$$= \mu e^{\mu(e^t - t - 1)}\left[2\mu e^{2t} + (1-2\mu)e^t + \mu(e^t - 1)\left\{\mu e^{2t} + (1-2\mu)e^t + \mu\right\}\right]_{t=0}$$

$$= \mu(1)\left[2\mu + (1-2\mu) + \mu(0)\{\text{———}\}\right]$$

$$= \mu(\cancel{2\mu} + 1 - \cancel{2\mu})$$

$$= \mu(1)$$

$$\Rightarrow \mu_3 = \mu$$

$$\mu_4 = \frac{d^4}{dt^4}\mu e^{\mu(e^t - t - 1)}\Big|_{t=0}$$

$$= \frac{d}{dt}\left[\mu e^{\mu(e^t - t - 1)}\left\{2\mu e^{2t} + (1-2\mu)e^t + \mu(e^t - 1)\left\{\mu e^{2t} + (1-2\mu)e^t + \mu\right\}\right]\Big|_{t=0}$$

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$$= \mu e^{\mu(e^t - t - 1)} \Big[4\mu e^{2t} + (1-2\mu)e^t + \mu(e^t-1)\{2\mu e^{2t} + (1-2\mu)e^t\} + \mu e^t \{\mu e^{2t} + (1-2\mu)e^t + \mu\} \Big|_{t=0} + \Big[\mu e^{\mu(e^t - t - 1)} \mu(e^t-1) \{2\mu e^{2t} + (1-2\mu)e^t + \mu(e^t-1)\{\mu e^{2t} + (1-2\mu)e^t + \mu\}\Big]_{t=0}$$

$$= \mu\{4\mu + 1 - 2\mu + \mu(\cancel{\mu} + 1 - \cancel{2\mu} + \cancel{\mu}) + \mu(0)\}$$

$$= \mu(2\mu + 1 + \mu(1))$$

$$= \mu(2\mu + 1 + \mu)$$

$$= \mu(3\mu + 1) \qquad = 3\mu^2 + \mu$$

A/2017

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\mu^2}{\mu^3} = \frac{1}{\mu}$$

A/2018

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\mu^2 + \mu}{\mu^2} = 3 + \frac{1}{\mu}$$

Recurrence Formula.

We know that

$$P(X = x) = \frac{\mu^x}{x!} e^{-\mu} \quad ; x = 0, 1, 2, \ldots \quad \text{——— (1)}$$

and

$$P(X = x-1) = \frac{\mu^{x-1}}{(x-1)!} e^{-\mu} \quad \text{——— (2)}$$

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Dividing ① by ②, we get

$$\frac{P(X=x)}{P(X=x-1)} = \frac{\mu^x}{x!} e^{-\mu} \times \frac{(x-1)!}{\mu^{x-1} e^{-\mu}}$$

$$= \frac{\cancel{\mu^x}}{x\cancel{(x-1)!}} \times \frac{\cancel{(x-1)!}}{\cancel{\mu^x} \cdot \mu^{-1}}$$

$$= \frac{\mu}{x}$$

$$\Rightarrow P(X=x) = \frac{\mu}{x} P(X=x-1) \quad ; x = 1, 2, \ldots$$

Theorem. 2013, 2014, 2016,

For poisson distribution, prove the relation

$$\mu_{r+1} = m\left(r\mu_{r-1} + \frac{d\mu_r}{dm}\right)$$

where μ_r is the rth moment about mean.

Hence find μ_2, μ_3 and μ_4.

Proof.

We know that, the rth moment about the mean is

$$\mu_r = E(X-m)^r$$

where m is the mean.

$$\Rightarrow \mu_r = \sum_{x=0}^{\infty} (x-m)^r \frac{m^x}{x!} e^{-m}$$

Diff. w.r.t. to m, we get

$$\frac{d\mu_r}{dm} = \sum_{x=0}^{\infty} (x-m)^r \frac{m^x}{x!} e^{-m} (-1)$$

$$+ \sum_{x=0}^{\infty} (x-m)^r \, x \frac{m^{x-1}}{x!} e^{-m} + r \sum_{x=0}^{\infty} (x-m)^{r-1} (-1) \frac{m^x}{x!} e^{-m}$$

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$$= -\sum_{x=0}^{\infty} (x-m)^{s} \frac{m^x}{x!} e^{-m} + \sum_{x=0}^{\infty} (x-m)^{s} \frac{x\, m^{x-1}}{x!} e^{-m}$$

$$+ s \sum_{x=0}^{\infty} (x-m)^{s-1} (-1) \frac{m^x}{x!} e^{-m}$$

$$= \sum_{x=0}^{\infty} (x-m)^{s} \frac{m^x}{x!} e^{-m} \left(-1 + \frac{x}{m}\right) - s\mu_{s-1}$$

$$= \sum_{x=0}^{\infty} (x-m)^{s} \frac{m^x}{x!} e^{-m} \left(\frac{-m+x}{m}\right) - s\mu_{s-1}$$

$$= \sum_{x=0}^{\infty} \frac{(x-m)^{s+1}}{m} \frac{m^x}{x!} e^{-m} - s\mu_{s-1}$$

$$\Rightarrow \frac{d\mu_s}{dm} + s\mu_{s-1} = \frac{1}{m}\mu_{s+1}$$

$$\Rightarrow \mu_{s+1} = m\left(s\mu_{s-1} + \frac{d\mu_s}{dm}\right)$$

Now, we find μ_2, μ_3, μ_4 as follows.

$$\mu_2 = m\left(1\mu_0 + \frac{d\mu_1}{dm}\right)$$

Since $\mu_0 = E(X^0) = 1$

and $\mu_1 = 0$

Therefore,

$$\mu_2 = m(1 + 0) = m$$

$$\mu_3 = m\left(2\mu_1 + \frac{d\mu_2}{dm}\right)$$

$$= m\left(0 + \frac{dm}{dm}\right)$$

$$= m(1) = m$$

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$$\mu_4 = m\left(3\mu_2 + \frac{d\mu_3}{dm}\right)$$

$$= m\left(3m + \frac{dm}{dm}\right)$$

$$= m(3m+1) = 3m^2 + m$$

Cumulant Generating Function. N/2013

The cumulant G.F is given by

$$K(t) = \ln M_{0}(t)$$

$$= \ln e^{\mu(e^t - 1)}$$

$$= \mu(e^t - 1)$$

$$= \mu\left[\cancel{1} + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots - \cancel{1}\right]$$

$$K(t) = \mu\left[\frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right]$$

The general expression for cumulant g.f is given by

$$K(t) = K_1\frac{t}{1!} + K_2\frac{t^2}{2!} + K_3\frac{t^3}{3!} + K_4\frac{t^4}{4!} + \cdots$$

Comparing, we get

$K_1 = \mu$; $K_3 = \mu$

$K_2 = \mu$; $K_4 = \mu$

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Hypergeometric Distribution

Formula.

$$P(X=x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}} \quad ; x = 0, 1, 2, \cdots n$$

where

N = Number of total items

n = number of selected items.

K = number of fixed items in N.

x = number of fixed items in n.

It is also denoted by $h(x; N, n, K)$

Mean.

S/2018 $E(X) = \sum_{x=0}^{n} x\, P(X=x)$

A/2012

$$= \sum_{x=0}^{n} x \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$

$$= \sum_{x=0}^{n} x \frac{K! \binom{N-K}{n-x}}{x!\,(K-x)!\,\binom{N}{n}}$$

$$= \frac{K!}{\binom{N}{n}} \sum_{x=0}^{n} \frac{\cancel{x}\binom{N-K}{n-x}}{\cancel{x}(x-1)!\,(K-x)!}$$

$$= \frac{K!}{\binom{N}{n}} \sum_{x=1}^{n} \frac{\binom{N-K}{n-x}}{(x-1)!\,(K-x)!}$$

$$= \frac{K(K-1)!\; n!\,(N-n)!}{N!} \sum_{x=1}^{n} \frac{\binom{N-K}{n-x}}{(x-1)!\,(K-x)!}$$

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$$= \frac{K\,n!\,(N-n)!}{N!} \sum_{x=1}^{n} \frac{(K-1)!}{(x-1)!\,(K-x)!} \binom{N-K}{n-x}$$

$$= \frac{K\,n!\,(N-n)!}{N!} \sum_{x=1}^{n} \binom{K-1}{x-1}\binom{N-K}{n-x}$$

Let $x-1=j \Rightarrow x=j+1$

$x=1 \Rightarrow j=0$

$x=n \Rightarrow j=n-1$

Then,

$$E(X) = \frac{K\,n!\,(N-n)!}{N!} \sum_{j=0}^{n-1} \binom{K-1}{j}\binom{N-K}{n-j-1}$$

$$= \frac{K\,n!\,(N-n)!}{N!} \sum_{j=0}^{n-1} \binom{K-1}{j}\binom{N-K}{n-1-j}$$

Using the property

$$\sum_{j=0}^{m} \binom{a}{j}\binom{b}{m-j} = \binom{a+b}{m}, \text{ we get}$$

$$E(X) = \frac{K\,n!\,(N-n)!}{N!} \binom{\cancel{K}-1+N-\cancel{K}}{\cancel{j}+n-1-\cancel{j}}$$

$$= \frac{K\,n!\,(N-n)!}{N!} \binom{N-1}{n-1}$$

$$= \frac{K\,n!\,\cancel{(N-n)!}}{N!} \frac{(N-1)!}{(n-1)!\,\cancel{(N-n)!}}$$

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$$= \frac{kn\,\cancel{(n-1)!}}{N\,\cancel{(N-1)!}} \quad \frac{\cancel{(N-1)!}}{\cancel{(n-1)!}}$$

$$= \frac{kn}{N} = n(k/N) = np$$

$$\Rightarrow \boxed{E(X) = np}$$

where $p = k/N$

Variance. S/2018 A/2012

$$V(X) = E(X^2) - (E(X))^2$$

Note that

$$E(X^2) = E(X(X-1)) + E(X).$$

Now,

$$E(X(X-1)) = \sum_{x=0}^{n} x(x-1) \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$$

$$= \sum_{x=0}^{n} x(x-1) \frac{k!}{x!\,(k-x)!} \frac{\binom{N-k}{n-x}}{\binom{N}{n}}$$

$$= \sum_{x=0}^{n} \cancel{x(x-1)} \frac{k(k-1)(k-2)!}{\cancel{x(x-1)}(x-2)!\,(k-x)!} \frac{\binom{N-k}{n-x}}{\binom{N}{n}}$$

$$= \sum_{x=2}^{n} k(k-1) \frac{(k-2)!}{(x-2)!\,(k-x)!} \frac{\binom{N-k}{n-x}}{\binom{N}{n}}$$

$$= k(k-1) \sum_{x=2}^{n} \binom{k-2}{x-2} \frac{\binom{N-k}{n-x}}{\binom{N}{n}}$$

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Let $x-2 = y$

$x=2 \Rightarrow y=0 \quad ; x=n \Rightarrow y=n-2$

Then

$$E(X(X-1)) = \frac{K(K-1)}{\binom{N}{n}} \sum_{y=0}^{n-2} \binom{K-2}{y}\binom{N-K}{n-2-y}$$

$$= \frac{K(K-1)\,n!\,(N-n)!}{N!} \sum_{y=0}^{n-2} \binom{K-2}{y}\binom{N-K}{n-2-y}$$

Using the property $\sum_{j=0}^{n} \binom{a}{j}\binom{b}{n-j} = \binom{a+b}{n}$

we get

$$E(X(X-1)) = \frac{K(K-1)\,n!\,(N-n)!}{N!} \binom{K-2+N-K}{y+n-2-y}$$

$$= \frac{K(K-1)\,n!\,(N-n)!}{N!} \binom{N-2}{n-2}$$

$$= \frac{K(K-1)\,n!\,\cancel{(N-n)!}}{N!} \cdot \frac{(N-2)!}{(n-2)!\,\cancel{(N-n)!}}$$

$$= \frac{K(K-1)\,n(n-1)\cancel{(n-2)!}}{N(N-1)\cancel{(N-2)!}} \cdot \frac{\cancel{(N-2)!}}{\cancel{(n-2)!}}$$

$$= \frac{K(K-1)\,n(n-1)}{N(N-1)}$$

Thus $V(X) = K(K-1)\dfrac{n(n-1)}{N(N-1)} + np - n^2p^2$

$$= K(K-1)\frac{n(n-1)}{N(N-1)} + \frac{nK}{N} - \frac{n^2K^2}{N^2}$$

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$$= \frac{K(K-1)n(n-1)N + nN(N-1)K - n^2K^2(N-1)}{N^2(N-1)}$$

$$= \frac{nK\{(K-1)(n-1)N + N(N-1) - nK(N-1)\}}{N^2(N-1)}$$

$$= \frac{nK\{(Kn-K-n+1)N + (N^2-N) - nKN + nK\}}{N^2(N-1)}$$

$$= \frac{nK\{\cancel{nKN} - KN - nN + \cancel{N} + N^2 - \cancel{N} - \cancel{nKN} + nK\}}{N^2(N-1)}$$

$$= \frac{nK(N^2 - NK + nK - nN)}{N^2(N-1)}$$

$$= \frac{nK\{N(N-K) + n(K-N)\}}{N^2(N-1)}$$

$$= \frac{nK\{N(N-K) - n(N-K)\}}{N^2(N-1)}$$

$$= \frac{nK(N-K)(N-n)}{N^2(N-1)}$$

$$= n\left(\frac{K}{N}\right)\left(\frac{N-K}{N}\right)\left(\frac{N-n}{N-1}\right)$$

$$= npq\,\frac{N-n}{N-1}$$

where $\frac{N-K}{N} = q$

and $K/N = p$

Hence,

$$\boxed{V(X) = npq\,\frac{N-n}{N-1}}$$

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Question.

If N becomes indefinitely large, the hypergeometric probability distribution tends to the binomial prob. distribution.

Sol.

The hypergeometric formula is given by

$$h(x; N, n, K) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}} \; ; x = 0, 1, 2, \cdots n$$

$$= \frac{K!}{x!\,(K-x)!} \cdot \frac{(N-K)!}{(n-x)!\,(N-K-n+x)!} \cdot \frac{n!\,(N-n)!}{N!}$$

$$= \frac{n!}{x!\,(n-x)!} \cdot \frac{K!\,(N-K)!}{(K-x)!\,(N-K-n+x)!} \cdot \frac{(N-n)!}{N!}$$

$$= \binom{n}{x} \frac{K(K-1)(K-2)\cdots(K-x+1)\cancel{(K-x)!}\,(N-K)!}{\cancel{(K-x)!}\,(N-K-n+x)!} \cdot \frac{(N-n)!}{N!}$$

$$= \binom{n}{x} \frac{K(K-1)(K-2)\cdots(K-x+1)(N-K)(N-K-1)(N-K-2)\cdots(N-K-n+x+1)\,\cancel{(N-K-n+x)!}}{\cancel{(N-K-n+x)!}\,(N-n)!} \Big| \frac{(N-n)!}{N!}$$

$$= \binom{n}{x} \frac{K(K-1)(K-2)\cdots(K-x+1)(N-K)(N-K-1)(N-K-2)\cdots(N-K-n+x+1)\,\cancel{(N-n)!}}{N(N-1)(N-2)(N-3)\cdots(N-n+1)\cancel{(N-n)!}}$$

$$= \binom{n}{x} \frac{K(K-1)(K-2)\cdots(K-x+1)(N-K)(N-K-1)\cdots(N-K-n+x+1)}{N(N-1)(N-2)\cdots(N-n+1)}$$

$$= \binom{n}{x} \frac{N^{x}\frac{K}{N}\left(\frac{K}{N}-\frac{1}{N}\right)\left(\frac{K}{N}-\frac{2}{N}\right)\cdots\left(\frac{K}{N}-\frac{x-1}{N}\right)N^{n-x}\left(1-\frac{K}{N}\right)\left(1-\frac{K+1}{N}\right)\cdots\left(1-\frac{K}{N}-\frac{n}{N}+\frac{x+1}{N}\right)}{N^{n}\,1\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\cdots\left(1-\frac{n-1}{N}\right)}$$

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As we know that $p = K/N$

So $$h(x;N,n,K) = \binom{n}{x} \frac{p(p-1/N)(p-2/N)\cdots(p-\frac{x-1}{N})(1-p)(1-p-1/N)\cdots(1-p-\frac{n}{N}-\frac{x-1}{N})}{1(1-1/N)(1-2/N)\cdots(1-\frac{n-1}{N})}$$

Taking $\lim_{N\to\infty}$, we get

$$\lim_{N\to\infty} h(x;N,n,K) = \binom{n}{x} \frac{\lim_{N\to\infty} p(p-1/N)(p-2/N)\cdots(p-\frac{x-1}{N})(1-p)(1-p-1/N)\cdots(1-p-\frac{n}{N}+\frac{x+1}{N})}{\lim_{N\to\infty} 1(1-1/N)(1-2/N)\cdots(1-\frac{n-1}{N})}$$

If N becomes indefinitely large, then

$\frac{x}{N}$, $\frac{n-x}{N}$ and $\frac{n}{N}$ each goes to zero.

$$\Rightarrow \lim_{N\to\infty} h(x;N,n,K) = \binom{n}{x} \frac{p(p)(p)\cdots(p)(1-p)(1-p)\cdots(1-p)}{(1)(1)(1)\cdots(1)(1)}$$

$$= \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \binom{n}{x} p^x q^{n-x}$$

$$= b(x;n,p)$$

$$\Rightarrow \lim_{N\to\infty} h(x;N,n,K) = b(x;n,p)$$

2016

Prove that

$$P(X=x+1) = \frac{(K-x)(n-x)}{(x+1)(N-K-n+x+1)} P(X=x)$$

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Negative Binomial Distribution.

Formula.

$$P(X=x) = \binom{x-1}{k-1} p^k q^{x-k} \quad ; x = k, k+1, k+2, \cdots$$

Let $y = x-k$.

Then

$$P(X=y) = \binom{y+k-1}{k-1} p^k q^y \quad ; y = 0, 1, 2, \cdots$$

or

$$P(X=x) = \binom{x+k-1}{k-1} p^k q^x \quad ; x = 0, 1, 2, \cdots$$

It is denoted by $b^*(x; k, p)$; where k and p are parameters.

Mean.

A/2018 $$E(X) = \sum_{x=0}^{\infty} x \binom{x+k-1}{k-1} p^k q^x$$

$$= p^k \left[\binom{k}{k-1} q + 2\binom{k+1}{k-1} q^2 + 3\binom{k+2}{k-1} q^3 + 4\binom{k+3}{k-1} q^4 + \cdots \right]$$

$$= p^k \left[\frac{k!}{(k-1)!\,(1)!} q + 2\frac{(k+1)!}{(k-1)!\,(2)!} q^2 + 3\frac{(k+2)!}{(k-1)!\,(3)!} q^3 + 4\frac{(k+3)!}{(k-1)!\,(4)!} q^4 + \cdots \right]$$

$$= p^k \left[\frac{k\cancel{(k-1)!}}{\cancel{(k-1)!}} q + \frac{(k+1)k\cancel{(k-1)!}}{\cancel{(k-1)!}} q^2 + \frac{(k+2)(k+1)k\cancel{(k-1)!}}{\cancel{(k-1)!}\,(2)} q^3 + \frac{(k+3)(k+2)(k+1)(k)\cancel{(k-1)!}}{\cancel{(k-1)!}\,(3)!} q^4 + \cdots \right]$$

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$$= p^k \left[kq + k(k+1)q^2 + \frac{k(k+1)(k+2)}{2!} q^3 + \frac{k(k+1)(k+2)(k+3)}{3!} q^4 + \cdots \right]$$

$$= p^k kq \left[1 + (k+1)q + \frac{(k+1)(k+2)}{2!} q^2 + \frac{(k+1)(k+2)(k+3)}{3!} q^3 + \cdots \right]$$

$$= p^k kq \left[1 + (-k-1)(-q) + \frac{(-k-1)(-k-2)}{2!} (-q)^2 + \frac{(-k-1)(-k-2)(-k-3)}{3!} (-q)^3 + \cdots \right]$$

$$= p^k kq \, (1-q)^{-k-1}$$

$$= p^k kq \, p^{-k-1} \qquad \because 1-q = p$$

$$= p^k kq \, p^{-k} \cdot p^{-1}$$

$$= \frac{kq}{p}$$

$$\Rightarrow \boxed{E(X) = \frac{kq}{p}}$$

Variance A/2018

$$V(X) = E(X^2) - (E(X))^2$$

Also, $E(X^2) = E(X(X-1)) + E(X)$.

Note that

$$E(X(X-1)) = \sum_{x=0}^{\infty} x(x-1) \binom{x+k-1}{k-1} p^k q^x$$

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$$= p^k\left[2(1)\binom{k+1}{k-1}q^2 + 3(2)\binom{k+2}{k-1}q^3 + 4(3)\binom{k+3}{k-1}q^4 + 5(4)\binom{k+4}{k-1}q^5 + \cdots\right]$$

$$= p^k\left[2\frac{(k+1)!}{(k-1)!\,(2)!}q^2 + 3(2)\frac{(k+2)!}{(k-1)!\,(3)!}q^3 + 4(3)\frac{(k+3)!}{(k-1)!\,(4)!}q^4 + 5(4)\frac{(k+4)!}{(k-1)!\,(5)!}q^5 + \cdots\right]$$

$$= p^k\left[\frac{(k+1)(k)\cancel{(k-1)!}}{\cancel{(k-1)!}}q^2 + \frac{(k+2)(k+1)(k)\cancel{(k-1)!}}{\cancel{(k-1)!}}q^3 + \frac{(k+3)(k+2)(k+1)(k)\cancel{(k-1)!}}{\cancel{(k-1)!}\;2!}q^4 + \frac{(k+4)(k+3)(k+2)(k+1)(k)\cancel{(k-1)!}}{\cancel{(k-1)!}\,(3)!}q^5 + \cdots\right]$$

$$= p^k k(k+1)q^2\left[1 + (k+2)q + \frac{(k+2)(k+3)}{2!}q^2 + \frac{(k+2)(k+3)(k+4)}{3!}q^3 + \cdots\right]$$

$$= k(k+1)q^2p^k\left[1 + (-k-2)(-q) + \frac{(-k-2)(-k-3)}{2!}(-q)^2 + \frac{(-k-2)(-k-3)(-k-4)}{3!}(-q)^3 + \cdots\right]$$

$$= k(k+1)q^2p^k(1-q)^{-k-2}$$

$$= k(k+1)q^2p^k p^{-k-2} \qquad \because 1-q=p$$

$$= k(k+1)q^2p^k p^{-k}p^{-2}$$

$$= k(k+1)\frac{q^2}{p^2}$$

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$$\Rightarrow \boxed{E(X(X-1)) = K(K+1)\frac{q^2}{p^2}}$$

$$E(X^2) = K(K+1)\frac{q^2}{p^2} + K\frac{q}{p} = \frac{K(K+1)q^2 + Kpq}{p^2}$$

Thus,

$$V(X) = \frac{K(K+1)q^2 + pqK}{p^2} - \frac{K^2q^2}{p^2}$$

$$= \frac{K(K+1)q^2 + pqK - K^2q^2}{p^2}$$

$$= \frac{\cancel{K^2q^2} + Kq^2 + pqK - \cancel{K^2q^2}}{p^2}$$

$$= \frac{Kq(q+p)}{p^2}$$

$$= \frac{Kq}{p^2} \qquad \because q+p=1$$

$$\Rightarrow \boxed{Var(X) = \frac{Kq}{p^2}}$$

Moments About Origin.

$$\mu_r' = E(X-0)^r$$
$$= E(X^r)$$

$$r=1 \Rightarrow \mu_1' = E(X) = Kq/p$$

$$r=2 \Rightarrow \mu_2' = E(X^2) = \frac{K(K+1)q^2 + Kpq}{p^2}$$

$$r=3 \Rightarrow \mu_3' = E(X^3)$$

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$= E(X(X-1)(X-2)) + 3E(X(X-1)) + E(X)$ — ①

Note that

$E(X(X-1)(X-2)) = K(K+1)(K+2)\, q^3/p^3$ After solving

Thus, equation ① becomes

$$\mu_3' = K(K+1)(K+2)\frac{q^3}{p^3} + 3K(K+1)\frac{q^2}{p^2} + \frac{Kq}{p}$$

$$= \frac{Kq}{p^3}\left\{(K+1)(K+2)q^2 + 3(K+1)pq + p^2\right\}.$$

$s=4 \Rightarrow \mu_4' = E(X^4)$

$$= E(X(X-1)(X-2)(X-3)) + 6E(X(X-1)(X-2)) + 7E(X(X-1)) + E(X).$$

We can easily find

$$E(X(X-1)(X-2)(X-3)) = K(K+1)(K+2)(K+3)\frac{q^4}{p^4}$$

$$\mu_4' = K(K+1)(K+2)(K+3)\frac{q^4}{p^4} + 6K(K+1)(K+2)\, q^3/p^3 + 7K(K+1)\, p^2/p^2 + \frac{Kq}{p}$$

$$= \frac{Kq}{p^4}\left\{(K+1)(K+2)(K+3)q^3 + 6(K+1)(K+2)q^2p + 7(K+1)p^2q + p^3\right\}.$$

Moments About Mean.

$$\mu_s = E(X-\mu)^s$$

where μ is mean.

$$\mu_1 = E(X) - \mu = \mu - \mu = 0$$

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$= \frac{K(K+1)q^2 + Kpq}{p^2} - \frac{K^2q^2}{p^2} = \frac{Kq}{p^2}$$ by variance calculation

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$$\mu_3^{\circ} = \mu_3' - 3\mu_1'\mu_1' + 2(\mu_1')^3$$

$$= \frac{kq}{p^3}\{(k+1)(k+2)q^2 + 3(k+1)pq + p^2\}$$

$$-3\left\{\frac{k(k+1)q^2 + kpq}{p^2}\right\}\left(\frac{kq}{p}\right) + 2\frac{k^3q^3}{p^3}$$

$$= \frac{kq}{p^3}\{(k+1)(k+2)q^2 + 3(k+1)pq + p^2 - 3k(k+1)q^2 - 3kpq + 2k^2q^2\}$$

$$= \frac{kq}{p^3}\{(k^2+3k+2)q^2 + (3k+3)pq + p^2 - 3(k^2+k)q^2 - 3kpq + 2k^2q^2\}$$

$$= \frac{kq}{p^3}\{\cancel{k^2q^2} + \cancel{3kq^2} + 2q^2 + \cancel{3kpq} + 3pq + p^2 - \cancel{3k^2q^2} - \cancel{3kq^2} - \cancel{3kpq} + \cancel{2k^2q^2}\}$$

$$= \frac{kq}{p^3}\{2q^2 + 3pq + p^2\}$$

$$= \frac{kq}{p^3}\{2q^2 + 2pq + pq + p^2\}$$

$$= \frac{kq}{p^3}\{2q(q+p) + p(q+p)\}$$

$$= \frac{kq}{p^3}\{(2q+p)(q+p)\}$$

$$= \frac{kq}{p^3}(q+q+p)$$

$$\Rightarrow \boxed{\mu_3 = \frac{kq}{p^3}(1+q)}$$

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$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$

$$= \frac{kq}{p^4}\{(k+1)(k+2)(k+3)q^3 + 6(k+1)(k+2)q^2p + 7(k+1)p^2q + p^3\}$$

$$-4\frac{kq}{p^3}\{(k+1)(k+2)q^2 + 3(k+1)pq + p^2\}\left(\frac{kq}{p}\right)$$

$$+6\left\{\frac{k(k+1)q^2 + kpq}{p^2}\right\}\left(\frac{kq}{p^2}\right)^2 - \frac{3k^4q^4}{p^4}$$

$$= \frac{kq}{p^4}\{(k^2+3k+2)(k+3)q^3 + 6(k^3+3k+2)q^2p + (7k+7)p^2q + p^3 - 4k(k^2+3k+2)q^3 - 12k(k+1)pq^2 - 4p^2kq + 6k^2(k+1)q^3 + 6k^2pq^2 - 3k^3q^3\}$$

$$= \frac{kq}{p^4}\{(k^3+3k^2+2k+3k^2+9k+6)q^3 + 6k^2q^2p + 18kq^2p + 12q^2p + 7kp^2q + 7p^2q + 7p^2q + p^3 - 4(k^3+3k^2+2k)q^3 - 12(k^2+k)pq^2 - 4p^2kq + 6(k^3+k^2)q^3 + 6k^2pq^2 - 3k^3q^3\}.$$

$$= \frac{kq}{p^4}\{\cancel{k^3q^3} + \cancel{6k^2q^3} + 11kq^3 + 6q^3 + \cancel{6k^2q^2p} + 18kq^2p + 12q^2p + 7kp^2q + 7p^2q + p^3 - \cancel{4k^3q^3} - \cancel{12k^2q^3} - 8kq^3 - \cancel{12k^2pq^2} - 12kpq^2 - 4p^2kq + \cancel{6k^3q^3} + \cancel{6k^2q^3} + \cancel{6k^2pq^2} - \cancel{3k^2q^3}\}$$

$$= \frac{kq}{p^4}\{3kq^3 + 6q^3 + 6kq^2p + 12q^2p + 3kp^2q + 7p^2q + p^3\}$$

$$= \frac{kq}{p^4}\{3kq(q^2+2pq+p^2) + q^2(6q+12p) + p^2(7q+p)\}$$

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$$= \frac{kq}{p^4}\left\{3kq(q+p)^2 + q^2(6+6p) + p^2(6q+1)\right\}$$

$$= \frac{kq}{p^4}\left\{3kq + 6q^2(1+p) + 6p^2q + p^2\right\}$$

$$= \frac{kq}{p^4}\left\{3kq + 6(q^2 + q^2p + p^2q) + p^2\right\}$$

$$= \frac{kq}{p^4}\left\{p^2 + 3kq + 6(q^2 + qp(q+p))\right\}$$

$$= \frac{kq}{p^4}\left\{p^2 + 3kq + 6(q^2 + qp)\right\}$$

$$= \frac{kq}{p^4}\left\{p^2 + 3kq + 6q(q+p)\right\}$$

$$= \frac{kq}{p^4}\left\{p^2 + 3kq + 6q\right\}.$$

$$\Rightarrow \boxed{\mu_4 = \frac{kq}{p^4}\left\{p^2 + 3q(k+2)\right\}}$$

M.G.F. About Origin.

$$M_0(t) = E(e^{tx})$$

$$= \sum_{x=0}^{\infty} e^{tx} \binom{x+k-1}{k-1} p^k q^x$$

$$= p^k \sum_{x=0}^{\infty} \binom{x+k-1}{k-1} (qe^t)^x$$

$$= p^k\left[\frac{(k-1)!}{(k-1)!\,0!}(qe^t)^0 + \binom{k}{k-1}(qe^t)^1 + \right.$$

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$$\binom{k+1}{k-1}(qe^t)^2 + \binom{k+2}{k-1}(qe^t)^3 + \cdots \Big]$$

$$= p^k\left[1 + \frac{k!}{(k-1)!\,(1)!}qe^t + \frac{(k+1)!}{(k-1)!\,(2)!}(qe^t)^2 + \frac{(k+2)!}{(k-1)!\,(3)!}(qe^t)^3 + \cdots\right]$$

$$= p^k\left[1 + k(qe^t) + \frac{(k+1)k}{2!}(qe^t)^2 + \frac{(k+2)(k+1)k}{3!}(qe^t)^3 + \cdots\right]$$

$$= p^k\left\{1 + (-k)(-qe^t) + \frac{(-k-1)(-k)}{2!}(-qe^t)^2 + \frac{(-k)(-k-1)(-k-2)}{3!}(-qe^t)^3 + \cdots\right]$$

$$= p^k(1-qe^t)^{-k}$$

$$\Rightarrow \boxed{M_0(t) = p^k(1-qe^t)^{-k}}$$

$$\mu_s' = \frac{d^s}{dt^s}M_0(t)\Big|_{t=0}$$

$$= \frac{d^s}{dt^s}p^k(1-qe^t)^{-k}\Big|_{t=0}$$

$$s=1 \Rightarrow \mu_1' = \frac{d}{dt}p^k(1-qe^t)^{-k}\Big|_{t=0}$$

$$= p^k(-k)(1-qe^t)^{-k-1}(-qe^t)\Big|_{t=0}$$

$$= p^k k q e^t (1-qe^t)^{-k-1}\Big|_{t=0}$$

$$= p^k k q (1-q)^{-k-1}$$

$$= p^k k q (p)^{-k-1}$$

$$= p^k k q p^{-k} p^{-1} = \frac{kq}{p}.$$

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$$r=2 \Rightarrow \mu_2' = \frac{d^2}{dt^2} p^k (1-qe^t)^{-k} \Big|_{t=0}$$

$$= \frac{d}{dt}\left[p^k k q e^t (1-qe^t)^{-k-1}\right]_{t=0}$$

$$= p^k k q \left\{ e^t (1-qe^t)^{-k-1} + e^t(-k-1)(1-qe^t)^{-k-2}(-qe^t)\right]_{t=0}$$

$$= p^k k q \left\{ e^t (1-qe^t)^{-k-1} + (k+1) q e^{2t} (1-qe^t)^{-k-2}\right\}_{t=0}$$

$$= p^k k q \left\{ (1-q)^{-k-1} + (k+1) q (1-q)^{-k-2}\right\}$$

$$= p^k k q \left\{ p^{-k} p^{-1} + (k+1) q p^{-k} p^{-2}\right\}$$

$$= p^k k q p^{-k} \left\{ \frac{1}{p} + \frac{(k+1)q}{p^2}\right\}$$

$$= \frac{kq}{p^2}\left\{ p + (k+1) q \right\}$$

$$= \frac{k(k+1)q^2 + kpq}{p^2}$$

$$\mu_3' = \frac{d^3}{dt^3} p^k (1-qe^t)^{-k} \Big|_{t=0}$$

$$= \frac{d}{dt}\left[p^k k q e^t (1-qe^t)^{-k-1} + p^k k q^2 (k+1) e^{2t} (1-qe^t)^{-k-2}\right]_{t=0}$$

$$= p^k k q \Big[e^t (1-qe^t)^{-k-1} + e^t(-k-1)(1-qe^t)^{-k-2}(-qe^t)$$
$$+ 2(k+1) q e^{2t} (1-qe^t)^{-k-2} +$$

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$$(K+1)qe^{2t}(-K-2)(1-qe^{t})^{-K-3}(-qe^{t})\Big|_{t=0}$$

$$= p^{K}Kq\Big[e^{t}(1-qe^{t})^{-K-1} + 3(K+1)qe^{2t}(1-qe^{t})^{-K-2} + (K+1)(K+2)q^{2}e^{3t}(1-qe^{t})^{-K-3}\Big]_{t=0}$$

$$= p^{K}Kq\Big[(1-q)^{-K-1} + 3(K+1)q(1-q)^{-K-2} + (K+1)(K+2)q^{2}(1-q)^{-K-3}\Big]$$

$$= p^{K}Kq\Big[p^{-K}p^{-1} + 3(K+1)qp^{-K}p^{-2} + (K+1)(K+2)q^{2}p^{-K}p^{-3}\Big]$$

$$= Kq\Big[\frac{1}{p} + \frac{3(K+1)q}{p^{2}} + \frac{(K+1)(K+2)q^{2}}{p^{3}}\Big]$$

$$= \frac{Kq}{p^{3}}\Big[(K+1)(K+2)q^{2} + 3(K+1)pq + p^{2}\Big]$$

$$\mu_4' = \frac{d^{4}}{dt^{4}}p^{K}(1-qe^{t})^{-K}\Big|_{t=0}$$

$$= \frac{d}{dt}p^{K}Kq\Big[e^{t}(1-qe^{t})^{-K-1} + 3(K+1)qe^{2t}(1-qe^{t})^{-K-2} + (K+1)(K+2)q^{2}e^{3t}(1-qe^{t})^{-K-3}\Big]_{t=0}$$

$$= p^{K}Kq\Big[e^{t}(1-qe^{t})^{-K-1} + e^{t}(-K-1)(1-qe^{t})^{-K-2}(-qe^{t}) + 6(K+1)qe^{2t}(1-qe^{t})^{-K-2} + 3(K+1)qe^{2t}(-K-2)(1-qe^{t})^{-K-3}(-qe^{t}) + 3(K+1)(K+2)q^{2}e^{3t}(1-qe^{t})^{-K-3} + (K+1)(K+2)q^{2}e^{3t}(-K-3)(1-qe^{t})^{-K-4}(-qe^{t})\Big]_{t=0}$$

$$= p^{K}Kq\Big[e^{t}(1-qe^{t})^{-K-1} + 7(K+1)qe^{2t}(1-qe^{t})^{-K-2} + 6(K+1)(K+2)q^{2}e^{3t}(1-qe^{t})^{-K-3} + (K+1)(K+2)(K+3)q^{3}e^{4t}(1-qe^{t})^{-K-4}\Big]_{t=0}$$

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$$= p^k k q \cdot [(1-q)^{-k-1} + 7(k+1) q (1-q)^{-k-2}$$
$$+ 6(k+1)(k+2) q^2 (1-q)^{-k-3} +$$
$$(k+1)(k+2)(k+3) q^3 (1-q)^{-k-4}]$$

$$= kq[p^{-1} + 7(k+1) q p^{-2} + 6(k+1)(k+2) q^2 p^{-3}$$
$$+ (k+1)(k+2)(k+3) q^3 p^{-4}]$$

$$= kq\left[\frac{1}{p} + \frac{7(k+1)q}{p^2} + \frac{6(k+1)(k+2)}{p^3} q^2 + \frac{(k+1)(k+2)(k+3)q^3}{p^4}\right]$$

$$\mu_4' = \frac{kq}{p^4}[(k+1)(k+2)(k+3)q^3 + 6(k+1)(k+2)pq^2 + 7(k+1)p^2 q + p^3].$$

M.G.F. About Mean.

$$M_\mu(t) = E(e^{t(x-\mu)})$$

where μ is mean.

$$\Rightarrow M_\mu(t) = \sum_{x=0}^{\infty} e^{t(x-\mu)} \binom{x+k-1}{k-1} p^k q^x$$

$$= \sum_{x=0}^{\infty} e^{tx} \cdot e^{-\mu t} \binom{x+k-1}{k-1} p^k q^x$$

$$= e^{-\mu t} \sum_{x=0}^{\infty} e^{tx} \binom{x+k-1}{k-1} p^k q^x$$

$$= e^{-\mu t} p^k (1-qe^t)^{-k}$$

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$$= p^{k} e^{-k\frac{q}{p}t} (1-qe^{t})^{-k}$$

$$= p^{k}\left[e^{\frac{q}{p}t}(1-qe^{t})\right]^{-k}$$

$$= p^{k}\left[e^{\frac{q}{p}t} - q\, e^{t(1+\frac{q}{p})}\right]^{-k}$$

$$= p^{k}\left[e^{\frac{q}{p}t} - q\, e^{\frac{(p+q)t}{p}}\right]^{-k}$$

$$\Rightarrow \boxed{M_{u}(t) = p^{k}\left[e^{\frac{q}{p}t} - q\, e^{\frac{1}{p}t}\right]^{-k}}$$

$$\mu_{s} = \frac{d^{s}}{dt^{s}} M_{u}(t)\Big|_{t=0}$$

$$= \frac{d^{s}}{dt^{s}} p^{k}\left(e^{\frac{q}{p}t} - q\, e^{\frac{1}{p}t}\right)^{-k}\Big|_{t=0}$$

$$\mu_{1} = \frac{d}{dt} p^{k}\left(e^{\frac{q}{p}t} - q\, e^{\frac{1}{p}t}\right)^{-k}\Big|_{t=0}$$

$$= p^{k}(-k)\left(e^{\frac{q}{p}t} - qe^{\frac{1}{p}t}\right)^{-k-1}\left(e^{\frac{q}{p}t}\frac{q}{p} - \frac{q}{p}e^{\frac{1}{p}t}\right)\Big|_{t=0}$$

$$= -kp^{k}(1-q)^{-k-1}\left(\frac{q}{p} - \frac{q}{p}\right) = 0$$

$$\mu_{2} = \frac{d^{2}}{dt^{2}} p^{k}\left(e^{\frac{q}{p}t} - qe^{\frac{1}{p}t}\right)^{-k}\Big|_{t=0}$$

$$= \frac{d}{dt}(-k)p^{k}\frac{q}{p}\left(e^{\frac{q}{p}t} - qe^{\frac{1}{p}t}\right)^{-k-1}\left(e^{\frac{q}{p}t} - e^{\frac{1}{p}t}\right)\Big|_{t=0}$$

$$= -kqp^{k-1}\Big[\left(e^{\frac{q}{p}t} - qe^{\frac{1}{p}t}\right)^{-k-1}\left(e^{\frac{q}{p}t}\frac{q}{p} - \frac{1}{p}e^{\frac{1}{p}t}\right)$$

$$(-k-1)\left(e^{q/p\,t} - qe^{1/p\,t}\right)^{-k-2}\left(e^{q/p\,t}\frac{q}{p} - \frac{q}{p}e^{1/p\,t}\right)$$

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$$\left(e^{q/p\,t} - e^{1/p\,t}\right)\Big]_{t=0}$$

$$= -kqp^{k-1}\Big[\left(e^{q/p\,t} - qe^{1/p\,t}\right)^{-k-1}\left(e^{\frac{q}{p}t}\frac{q}{p} - \frac{1}{p}e^{\frac{1}{p}t}\right)$$

$$-(k+1)\frac{q}{p}\left(e^{q/p\,t} - qe^{1/p\,t}\right)^{-k-2}\left(e^{q/p\,t} - e^{1/p\,t}\right)^2\Big]_{t=0}$$

$$= -kqp^{k-1}\left[(1-q)^{-k-1}\left(\frac{q}{p} - \frac{1}{p}\right) - (k+1)\frac{q}{p}(1-q)^{-k-2}(1-1)^2\right]$$

$$= -kqp^{k-1}\left[p^{-k-1}\left(\frac{q-1}{p}\right)\right]$$

$$= \frac{-kqp^{k}p^{-1}\,p^{-k}p^{-1}(q-1)}{p} = \frac{-kqp^{-2}(q-1)}{p}$$

$$= \frac{kq(1-q)}{p^3} = \frac{kqp}{p^3}$$

$$\Rightarrow \boxed{\mu_2 = \frac{kq}{p^2}}$$

$$\mu_3 = \frac{d^3}{dt^3}\, p^k\left(e^{\frac{q}{p}t} - qe^{\frac{1}{p}t}\right)^{-k}\Big|_{t=0}$$

$$= -kqp^{k-1}\frac{d}{dt}\Big[\left(e^{\frac{q}{p}t} - qe^{\frac{1}{p}t}\right)^{-k-1}\left(e^{\frac{q}{p}t}\frac{q}{p} - \frac{1}{p}e^{1/p\,t}\right)$$

$$-(k+1)\frac{q}{p}\left(e^{q/p\,t} - qe^{1/p\,t}\right)^{-k-2}\left(e^{q/p\,t} - e^{1/p\,t}\right)^2\Big]_{t=0}$$

$$= -kqp^{k-1}\Big[\left(e^{q/p\,t} - qe^{1/p\,t}\right)^{-k-1}\left(e^{q/p\,t}\frac{q^2}{p^2} - \frac{1}{p^2}e^{\frac{1}{p}t}\right)$$

$$+(-k-1)\left(e^{\frac{q}{p}t} - qe^{\frac{1}{p}t}\right)^{-k-2}\left(e^{\frac{q}{p}t}\frac{q}{p} - \frac{q}{p}e^{1/p\,t}\right)\left(e^{q/p\,t}\frac{q}{p} - \frac{1}{p}e^{\frac{1}{p}t}\right)$$

$$-(k+1)\frac{q}{p}\left(e^{q/p\,t} - qe^{1/p\,t}\right)^{-k-2}2\left(e^{q/p\,t} - e^{1/p\,t}\right)\left(e^{q/p\,t}\frac{q}{p} - \frac{1}{p}e^{1/p\,t}\right)$$

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$$+(K+1)(K+2)\frac{q}{p}\left(e^{q/p\,t}-qe^{1/p\,t}\right)^{-K-3}\left(e^{q/p\,t}\frac{q}{p}-\frac{q}{p}e^{1/p\,t}\right)\cdot$$

$$\left.\left(e^{q/p\,t}-e^{\frac{1}{p}t}\right)^2\right]_{t=0}$$

$$=-Kqp^{K-1}\Big[\left(e^{\frac{q}{p}t}-qe^{1/p\,t}\right)^{-K-1}\left(e^{\frac{q}{p}t}\frac{q^2}{p^2}-\frac{1}{p^2}e^{\frac{1}{p}t}\right)$$

$$-3\frac{q}{p}(K+1)\left(e^{q/p\,t}-qe^{1/p\,t}\right)^{-K-2}\left(e^{\frac{q}{p}t}-e^{\frac{1}{p}t}\right)\left(e^{\frac{q}{p}t}\frac{q}{p}-\frac{1}{p}e^{\frac{1}{p}t}\right)$$

$$\left.+(K+1)(K+2)\frac{q}{p}\left(e^{\frac{q}{p}t}-qe^{\frac{1}{p}t}\right)^{-K-3}\left(e^{q/p\,t}\frac{q}{p}-\frac{q}{p}e^{1/p\,t}\right)\left(e^{q/p\,t}-e^{1/p\,t}\right)^2\Big]_{t=0}$$

$$=-Kqp^{K-1}\left[(1-q)^{-K-1}\left(\frac{q^2}{p^2}-\frac{1}{p^2}\right)-0+0\right]$$

$$=-Kqp^{K-1}p^{-K-1}\frac{q^2-1}{p^2}$$

$$=\frac{-Kq}{p^4}(q+1)(q-1)$$

$$=\frac{Kq}{p^4}(1+q)(1-q)=\frac{KqP}{p^4}(1+q)$$

$$\Rightarrow \boxed{\mu_3=\frac{Kq}{p^3}(1+q)}$$

$$\mu_4=\frac{d^4}{dt^4}p^K\left(e^{q/p\,t}-qe^{1/p\,t}\right)^{-K}\Big|_{t=0}$$

$$=\frac{d}{dt}\Big[-Kqp^{K-1}\Big\{\left(e^{q/p\,t}-qe^{1/p\,t}\right)^{-K-1}\left(e^{q/p\,t}\frac{q^2}{p^2}-\frac{1}{p^2}e^{1/p\,t}\right)$$

$$-3\frac{q}{p}(K+1)\left(e^{q/p\,t}-qe^{1/p\,t}\right)^{-K-2}\left(e^{q/p\,t}-e^{1/p\,t}\right)\left(e^{q/p\,t}\frac{q}{p}-\frac{1}{p}e^{1/p\,t}\right)$$

$$\left.+(K+1)(K+2)\frac{q^2}{p^2}\left(e^{q/p\,t}-qe^{1/p\,t}\right)^{-K-3}\left(e^{q/p\,t}-e^{1/p\,t}\right)^3\Big\}\right]_{t=0}$$

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$$= -kqp^{k-1}\Big[(e^{q/p\,t} - qe^{1/p\,t})^{-k-1} \left(e^{q/p\,t}\frac{q^3}{p^3} - \frac{1}{p^3}e^{1/p\,t}\right)$$

$$+(-k-1)(e^{q/p\,t} - qe^{1/p\,t})^{-k-2}\left(e^{q/p\,t}\frac{q}{p} - \frac{q}{p}e^{1/p\,t}\right)\left(e^{q/p\,t}\frac{q^2}{p^2} - \frac{1}{p^2}e^{1/p\,t}\right)$$

$$-3(k+1)\frac{q}{p}(e^{q/p\,t} - qe^{1/p\,t})^{-k-2}(e^{q/p\,t} - e^{1/p\,t})\left(e^{q/p\,t}\frac{q^2}{p^2} - \frac{1}{p^2}e^{1/p\,t}\right)$$

$$-3(k+1)\,q/p\,(e^{q/p\,t} - qe^{1/p\,t})^{-k-2}\left(e^{q/p\,t}\frac{q}{p} - \frac{1}{p}e^{1/p\,t}\right)^2 +$$

$$3(k+1)(k+2)(e^{q/p\,t} - qe^{1/p\,t})^{-k-3}\left(e^{q/p\,t}\frac{q}{p} - \frac{q}{p}e^{1/p\,t}\right)$$

$$(e^{q/p\,t} - e^{1/p\,t})\left(e^{q/p\,t}\frac{q}{p} - \frac{1}{p}e^{1/p\,t}\right) +$$

$$(k+1)(k+2)\frac{q^2}{p^2}(e^{q/p\,t} - qe^{1/p\,t})^{-k-3}\,3(e^{q/p\,t} - e^{+1/p\,t})^2\left(e^{q/p\,t}\frac{q}{p} - \frac{1}{p}e^{1/p\,t}\right)$$

$$+(k+1)(k+1)\frac{q^2}{p^2}(-k-3)(e^{q/p\,t} - qe^{1/p\,t})^{-k-4}\left(e^{q/p\,t}\frac{q}{p} - \frac{q}{p}e^{1/p\,t}\right)$$

$$(e^{q/p\,t} - e^{1/p\,t})^3\Big]_{t=0}$$

$$= -kqp^{+k-1}\Big[(1-q)^{-k-1}\left(\frac{q^3}{p^3} - \frac{1}{p^3}\right) - 3(k+1)\frac{q}{p}(1-q)^{-k-2}\left(\frac{q}{p} - \frac{1}{p}\right)^2\Big]$$

$$= -kqp^{k-1}\Big[p^{-k-1}\frac{q^3-1}{p^3} - 3(k+1)\frac{q}{p}p^{-k-2}\frac{(q-1)^2}{p^2}\Big]$$

$$= +\frac{kq}{p}\Big[\frac{-q^3+1}{p^4} + \frac{3(k+1)q}{p^3}\frac{p^2}{p^2}\Big]$$

$$= \frac{kq}{p^4}\Big[\frac{1-q^3}{p} + 3(k+1)q\Big]$$

$$= \frac{kq}{p^4}\Big[\frac{(1-q)(1+q+q^2)}{p} + 3(k+1)q\Big]$$

$$= \frac{kq}{p^4}\Big[q^2+q+1 + 3(k+1)q\Big]$$

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$$= \frac{kq}{p^4}\left[1+p^2-2p+q+1+3(k+1)q\right]$$

$$= \frac{kq}{p^4}\left[p^2+2-2p+q+3(k+1)q\right]$$

$$= \frac{kq}{p^4}\left[p^2+2(1-p)+q+3(k+1)q\right]$$

$$= \frac{kq}{p^4}\left[p^2+3q+3(k+1)q\right]$$

$$\Rightarrow \boxed{\mu_4 = \frac{kq}{p^4}\left[p^2+3(k+2)q\right]}$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{k^2q^2}{\cancel{p^6}}(1+q)^2 \times \frac{\cancel{p^6}}{k^3q^3} = \frac{(1+q)^2}{kq}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{kq}{\cancel{p^4}}\left[p^2+(3k+6)q\right] \times \frac{\cancel{p^4}}{k^2q^2}$$

$$= \frac{p^2+3(k+2)q}{kq}$$

Theorem.

For the negative binomial distribution, prove the relation

$$\mu_{r+1} = q\left(\frac{\partial \mu_r}{\partial q} + \frac{rk}{p^2}\mu_{r-1}\right)$$

where μ_r is the rth moment about mean and hence find μ_2, μ_3, μ_4

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Proof.

We know that the rth moment about the mean is

$$\mu_r = E(X-\mu)^r \quad \text{, where } \mu \text{ is mean.}$$

$$= \sum_{x=0}^{\infty} (x-\mu)^r \binom{x+k-1}{k-1} p^k q^x$$

$$= \sum_{x=0}^{\infty} \left(x - \frac{kq}{p}\right)^r \binom{x+k-1}{k-1} p^k q^x$$

$$= \sum_{x=0}^{\infty} \left(x - \frac{kq}{1-q}\right)^r \binom{x+k-1}{k-1} (1-q)^k q^x.$$

Diff. partially, w.r.t. q, we get

$$\frac{\partial \mu_r}{\partial q} = \sum_{x=0}^{\infty} \left(x - \frac{kq}{1-q}\right)^r \binom{x+k-1}{k-1} (1-q)^k x q^{x-1}$$

$$+ \sum_{x=0}^{\infty} \left(x - \frac{kq}{1-q}\right)^r \binom{x+k-1}{k-1} k(1-q)^{k-1}(-1) q^x$$

$$+ \sum_{x=0}^{\infty} r\left(x - \frac{kq}{1-q}\right)^{r-1} \left(\frac{-k}{(1-q)^2}\right) \binom{x+k-1}{k-1} (1-q)^k q^x$$

$$= \sum_{x=0}^{\infty} \left(x - \frac{kq}{p}\right)^r \binom{x+k-1}{k-1} p^k q^x \left(\frac{x}{q} - \frac{k}{p}\right)$$

$$- \frac{kr}{p^2} \sum_{x=0}^{\infty} \left(x - \frac{kq}{p}\right)^{r-1} \binom{x+k-1}{k-1} p^k q^x$$

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$$= \sum_{x=0}^{\infty} \left(x - \frac{kq}{p}\right)^{r} \binom{x+k-1}{k-1} p^{k} q^{x} \left(\frac{px - kq}{pq}\right)$$

$$- \frac{rk}{p^2} \mu_{r-1}$$

$$\Rightarrow \frac{\partial \mu_r}{\partial q} + \frac{rk}{p^2} \mu_{r-1} = \frac{1}{q} \sum_{x=0}^{\infty} \left(x - \frac{kq}{p}\right)^{r} \binom{x+k-1}{k-1} p^{k} q^{x} \left(x - \frac{kq}{p}\right)$$

$$= \frac{1}{q} \sum_{x=0}^{\infty} \left(x - \frac{kq}{p}\right)^{r+1} \binom{x+k-1}{k-1} p^{k} q^{x}$$

$$= \frac{1}{q} \mu_{r+1}$$

$$\Rightarrow \mu_{r+1} = q\left(\frac{\partial \mu_r}{\partial q} + \frac{rk}{p^2} \mu_{r-1}\right)$$

Now we find μ_2, μ_3 and μ_4 as follows.

$$\mu_0 = E(X-\mu)^0 = 1$$

$$\mu_1 = q\left(\frac{\partial \mu_0}{\partial q} + 0\right) = 0$$

$$\mu_2 = q\left(\frac{\partial \mu_1}{\partial q} + \frac{k}{p^2} \mu_0\right) = q\left(0 + \frac{k}{p^2}(1)\right)$$

$$= kq/p^2.$$

$$\mu_3 = q\left(\frac{\partial \mu_2}{\partial q} + \frac{2k}{p^2} \mu_1\right) = q\left(\frac{\partial}{\partial q} \frac{kq}{p^2} + 0\right)$$

$$= qk \frac{\partial}{\partial q} \frac{q}{(1-q)^2} = kq\left[\frac{(1-q)^2(1) - 2(1-q)(-1)q}{(1-q)^4}\right]$$

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$$= \frac{kq}{p^4}\left[p^2 + 2pq\right] = \frac{kq}{p^3}(p+2q)$$

$$= \frac{kq}{p^3}(p+q+q)$$

$$\Rightarrow \mu_3 = \frac{kq}{p^3}(1+q) \qquad \because p+q=1$$

$$\mu_4 = q\left(\frac{\partial \mu_3}{\partial q} + 3\frac{k}{p^2}\mu_2\right)$$

$$= q\left[\frac{\partial}{\partial q}\left\{\frac{kq}{p^3}(1+q)\right\} + 3\frac{k}{p^2}\cdot\frac{kq}{p^2}\right]$$

$$= kq\left[\frac{\partial}{\partial q}\left\{\frac{q+q^2}{(1-q)^3}\right\} + 3\frac{kq}{p^4}\right]$$

$$= kq\left[\frac{(1-q)^3(1+2q) - (q+q^2)3(1-q)^2(-1)}{(1-q)^6} + \frac{3kq}{p^4}\right]$$

$$= kq\left[\frac{p^3(1+2q) + 3(q+q^2)p^2}{p^6} + \frac{3kq}{p^4}\right]$$

$$= \frac{kq}{p^4}\left[p(1+2q) + 3(q+q^2) + 3kq\right]$$

$$= \frac{kq}{p^4}\left[p + 2pq + 3q + 3q^2 + 3kq\right]$$

$$= \frac{kq}{p^4}\left[p + 2p(1-p) + 3q + 3(1-p)^2 + 3kq\right]$$

$$= \frac{kq}{p^4}\left[p + 2p - 2p^2 + 3q + 3 + 3p^2 - 6p + 3kq\right]$$

$$= \frac{kq}{p^4}\left[p^2 + 3kq + 3 - 3p + 3q\right]$$

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$$= \frac{kq}{p^4}\left[p^2 + 3kq + 3q + 3q\right]$$

$$= \frac{kq}{p^4}\left[p^2 + 3kq + 6q\right].$$

$$\Rightarrow \mu_4 = \frac{kq}{p^4}\left[p^2 + 3(k+2)q\right].$$

Question.

Prove that the negative binomial dis. can be reduced in the form

$$P(X=x) = \binom{-k}{x} p^k (-q)^x ; \; x = 0, 1, 2, \cdots$$

Sol.

We know that

$$P(X=x) = \binom{x+k-1}{k-1} p^k q^x \quad ; x = 0, 1, 2, \cdots$$

Since $\binom{n}{r} = \binom{n}{n-r}$

Therefore,

$$P(X=x) = \binom{x+k-1}{(x+k-1)-(k-1)} p^k q^x$$

$$= \binom{x+k-1}{x} p^k q^x$$

$$= \frac{(x+k-1)!}{x!\,(x+k-1-x)!} p^k q^x$$

$$= \frac{(x+k-1)!}{x!\,(k-1)!} p^k q^x$$

$$= \frac{(x+k-1)(x+k-2)(x+k-3)\cdots(k+1)k\cancel{(k-1)!}}{x!\,\cancel{(k-1)!}}$$

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$$= \frac{(x+k-1)(x+k-2)\cdots(k+1)k}{x!} p^k q^x$$

$$= \frac{(-1)^x(-k)(-k-1)\cdots(-k-x+2)(-k-x+1)}{x!} p^k q^x$$

$$= \frac{-k(-k-1)\cdots(-k-x+2)(-k-x+1)(-k-x)!}{x!\,(-k-x)!} p^k(-q)^x$$

$$= \frac{(-k)!}{x!\,(-k-x)!} p^k(-q)^x$$

$$= \binom{-k}{x} p^k(-q)^x \quad ; x = 0, 1, 2, \ldots$$

Cumulant G. Function.

The cumulant generating function is given by

$$K(t) = \ln M_0(t)$$

$$= \ln p^k(1-qe^t)^{-k}$$

$$= \ln\left[(p^{-1})^{-k}(1-qe^t)^{-k}\right]$$

$$= \ln\left[\left(\frac{1}{p}\right)^{-k}(1-qe^t)^{-k}\right]$$

$$= \ln\left(\frac{1}{p} - \frac{qe^t}{p}\right)^{-k}$$

$$= -k\ln\left(\frac{1}{p} - \frac{q}{p}e^t\right)$$

$$= -k\ln\left(\frac{1}{p} - \frac{1-p}{p}e^t\right)$$

$$= -k\ln\left(\frac{1}{p} - \frac{1}{p}e^t + 1e^t\right)$$

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$$= -K \ln\left[\frac{1}{p} - \left(\frac{1}{p} - 1\right) e^{t}\right]$$

$$= -K \ln\left[\frac{1}{p} - \left(\frac{1}{p} - 1\right)\left(1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right)\right]$$

$$= -K \ln\left[\frac{1}{p} - \left(\frac{1}{p} - 1\right) - \left(\frac{1}{p} - 1\right)\left(\frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right)\right]$$

$$= -K \ln\left[\frac{1}{p} - \frac{1}{p} + 1 - \frac{q}{p}\left(\frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right)\right]$$

$$= -K \ln\left(1 - \frac{q}{p}\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right)\right)$$

Since $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$

Therefore,

$$K(t) = +K\left[\frac{q}{p}\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right) + \frac{q^2}{2p^2}\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)^2 + \frac{q^3}{3p^3}\left(t + \frac{t^2}{2!} + \cdots\right)^3 + \frac{q^4}{4p^4}\left(t + \frac{t^2}{2!} + \cdots\right)^4 + \cdots\right]$$

$$= K\left[\frac{q}{p}\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right) + \frac{q^2}{2p^2}\left(t^2 + \frac{t^4}{4} + \frac{2t^3}{2!} + \frac{2t^4}{3!} + \cdots\right) + \frac{q^3}{3p^3}\left(t^3 + \frac{3t^4}{2!} + \cdots\right) + \frac{q^4}{4p^4}\left(t^4 + \cdots\right) + \cdots\right]$$

$$= K\left[\frac{q}{p}\frac{t}{1!} + \left(\frac{q}{p} + \frac{q^2}{p^2}\right)\frac{t^2}{2!} + \left(\frac{q}{p} + \frac{3q^2}{p^2} + \frac{2q^3}{p^3}\right)\frac{t^3}{3!} + \left(\frac{q}{p} + \frac{7q^2}{p^2} + \frac{12q^3}{p^3} + \frac{6q^4}{p^4}\right)\frac{t^4}{4!} + \cdots\right] \quad \text{——— (1)}$$

The general expression for cumulant G.F. is

$$K(t) = K_1\frac{t}{1!} + K_2\frac{t^2}{2!} + K_3\frac{t^3}{3!} + K_4\frac{t^4}{4!} + \cdots \quad \text{——— (2)}$$

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Comparing ① and ②, we get

$$K_1 = Kq/p$$

$$K_2 = K(q/p + q^2/p^2)$$

$$K_3 = K\left(\frac{q}{p} + \frac{3q^2}{p^2} + \frac{2q^3}{p^3}\right)$$

$$K_4 = K\left(\frac{q}{p} + \frac{7q^2}{p^2} + \frac{12q^3}{p^3} + \frac{6q^4}{p^4}\right)$$

$$\Rightarrow \boxed{K_1 = Kq/p}$$

$$\boxed{K_2 = \frac{Kq}{p^2}(p+q) = Kq/p^2}$$

$$\boxed{K_3 = \frac{Kq}{p^3}(p^2 + 3qp + 2q^2)}$$

$$= \frac{Kq}{p^3}\left[p^2 + pq + 2pq + 2q^2\right]$$

$$= \frac{Kq}{p^3}\left(p(p+q) + 2q(p+q)\right)$$

$$= \frac{Kq}{p^3}(p+2q) = \frac{Kq}{p^3}(1+q)$$

and

$$K_4 = \frac{Kq}{p^4}\left[p^3 + 7qp^2 + 12q^2p + 6q^3\right]$$

$$= \frac{Kq}{p^4}\left[p^2(p+7q) + 6q^2(2p+q)\right]$$

$$= \frac{Kq}{p^4}\left[p^2(p+q) + 6p^2q + 6q^2(2p+q)\right]$$

$$= \frac{Kq}{p^4}\left[p^2 + 6pq(p^2 + 2qp + q^2)\right]$$

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$$= \frac{kq}{p^4}\left[p^2 + 6q(q+p)^2\right]$$

$$\Rightarrow \boxed{K_4 = \frac{kq}{p^4}\left(p^2 + 6q\right)}$$

Geometric Distribution.

Formula

$$P(X=x) = pq^{x-1} \quad ; x = 1, 2, \cdots \infty$$

It is denoted by $g(x;p)$ where p is parameter.

Mean.

$$E(X) = \sum_{x=1}^{\infty} x p q^{x-1}$$

$$= p \sum_{x=1}^{\infty} x q^{x-1}$$

$$= p\{1 + 2q + 3q^2 + 4q^3 + \cdots\}.$$

$$= p\{1 + (+2)(+q) + (+3)(+q)^2 + 4q^3 + \cdots]$$

$$= p(1-q)^{-2} = p\,p^{-2} = \frac{1}{p}$$

$$\Rightarrow \boxed{E(X) = 1/p}$$

Variance.

$$V(X) = E(X^2) - (E(X))^2$$

Also. $E(X^2) = E(X(X-1)) + E(X)$.

Note that

$$E(X(X-1)) = \sum_{x=1}^{\infty} x(x-1)\, p q^{x-1}$$

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$$= p\{(1)(0)(1) + 2(1)q + 3(2)q^2 + 4(3)q^3 + 5(4)q^4 + \cdots\}$$

$$= p\{2q + 6q^2 + 12q^3 + 20q^4\}.$$

$$= 2pq\{1 + 3q + 6q^2 + 10q^3 + \cdots\}.$$

$$= 2pq(1-q)^{-3} = 2pq\,p^{-3}$$

$$= 2p^{-2}q = \frac{2q}{p^2}$$

Thus,

$$V(X) = \frac{2q}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{2q+p-1}{p^2}$$

$$= \frac{q+q+p-1}{p^2} = \frac{q+1-1}{p^2}$$

$$\Rightarrow \boxed{V(X) = q/p^2}$$

Moments About Origin.

$$\mu_r' = E(X^r)$$

$$r=1 \Rightarrow \mu_1' = E(X) = 1/p$$

$$r=2 \Rightarrow \mu_2' = E(X^2) = \frac{2q}{p^2} + \frac{1}{p}$$

$$= \frac{2q+p}{p^2} = \frac{q+q+p}{p^2}$$

$$\Rightarrow \boxed{\mu_2' = \frac{1+q}{p^2}}$$

$$\mu_3' = E(X^3) = E(X(X-1)(X-2)) + 3E(X(X-1)) + E(X)$$

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Note that

$$E(X(X-1)(X-2)) = \sum_{x=1}^{\infty} x(x-1)(x-2)\, p\, q^{x-1}$$

$$= p\{1(0)(-1)q^0 + 2(1)(0)q + 3(2)(1)q + 4(3)(2)q^3 + 5(4)(3)q^4 + 6(5)(4)q^5 + \cdots$$

$$= p\{6q^2 + 24q^3 + 60q^4 + 120q^5\}.$$

$$= 6pq^2(1 + 4q + 10q^2 + 20q^3 + \cdots)$$

$$= 6pq^2(1-q)^{-4}$$

$$= 6pq^2 p^{-4} = \frac{6q^2}{p^3}$$

Thus,

$$\mu_3' = \frac{6q^2}{p^3} + \frac{6q}{p^2} + \frac{1}{p} = \frac{6q^2 + 6pq + p^2}{p^3}$$

$$= \frac{6(1+p^2-2p) + 6p(1-p) + p^2}{p^3}$$

$$= \frac{6 + \cancel{6p^2} - 12p + 6p - \cancel{6p^2} + p^2}{p^3}$$

$$\Rightarrow \boxed{\mu_3' = \frac{p^2 - 6p + 6}{p^3}}$$

$$\mu_4' = E(X^4) = E(X(X-1)(X-2)(X-3)) + 6E(X(X-1)(X-2)) + 7E(X(X-1)) + E(X) \quad \text{——— (1)}$$

Note that

$$E(X(X-1)(X-2)(X-3)) = \sum_{x=1}^{\infty} x(x-1)(x-2)(x-3)\, p\, q^{x-1}$$

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$$= p\{4(3)(2)(1)q^3 + 5(4)(3)(2)q^4 + 6(5)(4)(3)q^5 + 7(6)(5)(4)q^6 + \cdots\}.$$

$$= p\{24q^3 + 5(24)q^4 + 15(24)q^5 + 35(24)q^6 + \cdots\}$$

$$= 24pq^3\{1 + 5q + 15q^2 + 35q^3 + \cdots\}$$

$$= 24pq^3(1-q)^{-5} = 24pq^3p^{-5} = 24\frac{q^3}{p^4}$$

Hence

$$① \Rightarrow \mu_4' = 24\frac{q^3}{p^4} + 36\frac{q^2}{p^3} + 14\frac{q}{p^2} + 1/p$$

$$= \frac{24q^3 + 36q^2p + 14qp^2 + p^3}{p^4}$$

$$= \frac{1}{p^4}\{24(1-p^3-3p+3p^2) + 36(1+p^2-2p)p + 14(1-p)p^2 + p^3\}.$$

$$= \frac{1}{p^4}\{24 - 24p^3 - 72p + \cancel{72p^2} + 36p + 36p^3 - \cancel{72p^2} + 14p^2 - 14p^3 + p^3\}$$

$$\Rightarrow \boxed{\mu_4' = \frac{1}{p^4}\{-p^3 + 14p^2 - 36p + 24\}}$$

Moments About Mean.

$$\mu_r = E(X-\mu)^r$$

where μ is the mean.

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$= \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{1+q-1}{p^2} = q/p^2$$

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$$\Rightarrow \boxed{\mu_2 = q/p^2}$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$

$$= \frac{p^2 - 6p + 6}{p^3} - 3\frac{(1+q)}{p^2}\frac{1}{p} + 2\left(\frac{1}{p^3}\right)$$

$$= \frac{p^2 - 6p + 6 - 3 - 3q + 2}{p^3}$$

$$= \frac{p^2 - 6p - 3q + 5}{p^3} = \frac{p^2 - 3(2p+q) + 5}{p^3}$$

$$= \frac{p^2 - 3(1+p) + 5}{p^3} = \frac{p^2 - 3 - 3p + 5}{p^3}$$

$$= \frac{p^2 - 3p + 2}{p^3} = \frac{p^2 - p - 2p + 2}{p^3}$$

$$= \frac{p(p-1) - 2(p-1)}{p^3} = \frac{(p-1)(p-2)}{p^3}$$

$$= \frac{(1-p)(2-p)}{p^3} = \frac{q}{p^3}(1+1-p) = \frac{q}{p^3}(1+q)$$

$$\Rightarrow \boxed{\mu_3 = \frac{q}{p^3}(1+q)}$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1)^4$$

$$= \frac{24q^3 + 36q^2p + 14qp^2 + p^3}{p^4} - 4\left\{\frac{6q^2 + 6pq + p^2}{p^3}\right\}\frac{1}{p}$$

$$+ 6\frac{(1+q)}{p^2}\left(1/p^2\right) - 3/p^4$$

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$$= \frac{24q^3+36q^2p+14qp^2+p^3-24q^2-24pq-4p^2+6+6q-3}{p^4}$$

$$= \frac{24q^3+36q^2p+14qp^2+p^3-24q^2-24pq-4p^2+6q+3}{p^4}$$

M.G.F About Origin.

$$M_0(t) = E(e^{tX})$$

$$= \sum_{x=1}^{\infty} e^{tx} p q^{x-1}$$

$$= \sum_{x=1}^{\infty} e^{tx} e^{t} e^{-t} p q^{x-1}$$

$$= \sum_{x=1}^{\infty} p e^{t} e^{t(x-1)} q^{x-1}$$

$$= p e^{t} \sum_{x=1}^{\infty} e^{t(x-1)} q^{x-1} = p e^{t} \sum_{x=1}^{\infty} (eq)^{x-1}$$

$$= p e^{t} \{ (e^{t} q)^{0} + (e^{t} q)^{1} + (e^{t} q)^{2} + (e^{t} q)^{3} + \cdots \}$$

$$= p e^{t} \{ 1 + q e^{t} + (q e^{t})^{2} + (q e^{t})^{3} + \cdots \}$$

$$= p e^{t} (1 - q e^{t})^{-1}$$

$$= p e^{t} (e^{t} (e^{-t} - q))^{-1}$$

$$= p e^{t} e^{-t} (e^{-t} - q)^{-1}$$

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$$\boxed{M_0(t) = p(e^{-t} - q)^{-1}}$$

$$\mu_r' = \frac{d^r}{dt^r} p(e^{-t} - q)^{-1} \Big|_{t=0}$$

$$\mu_1' = \frac{d}{dt} p(e^{-t} - q)^{-1} \Big|_{t=0}$$

$$= p(-1)(e^{-t} - q)^{-2}(-e^{-t}) \Big|_{t=0}$$

$$= p e^{-t}(e^{-t} - q)^{-2} \Big|_{t=0}$$

$$= p(1-q)^{-2} = p(p)^{-2} = 1/p$$

$$\mu_2' = \frac{d}{dt} p e^{-t}(e^{-t} - q)^{-2} \Big|_{t=0}$$

$$= p\left\{e^{-t}\left(2e^{-t}(e^{-t} - q)^{-3}\right) + (e^{-t} - q)^{-2}(e^{-t})\right\}_{t=0}$$

$$= p\left\{2(1-q)^{-3} - (1-q)^{-2}\right\}$$

$$= p\left\{\frac{2}{p^3} - \frac{1}{p^2}\right\} = p\left\{\frac{2-p}{p^3}\right\}$$

$$= \left\{\frac{2-p}{p^2}\right\} = \frac{1+1-p}{p^2} = \frac{1+q}{p^2}$$

$$\mu_3' = p\frac{d}{dt}\left\{2e^{-2t}(e^{-t} - q)^{-3} - e^{-t}(e^{-t} - q)^{-2}\right\}_{t=0}$$

$$= 2p\left\{(-3)(e^{-t} - q)^{-4}(-e^{-t})e^{-2t} + (-2e^{-2t})(e^{-t} - q)^{-3}\right\}_{t=0} - p\left\{-2(e^{-t} - q)^{-3}(-e^{-t})e^{t} - e^{-t}(e^{t} - q)^{-2}\right\}_{t=0}$$

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$$= \Big[2p\{3e^{-3t}(e^{-t}-q)^{-4} - 2e^{-2t}(e^{-t}-q)^{-3}\}$$
$$-p\{2e^{-2t}(e^{-t}-q)^{-3} - e^{-t}(e^{-t}-q)^{-2}\}\Big]_{t=0}$$

$$= p\Big[6e^{-3t}(e^{-t}-q)^{-4} - 4e^{-2t}(e^{-t}-q)^{-3}$$
$$-2e^{-2t}(e^{-t}-q)^{-3} + e^{-t}(e^{-t}-q)^{-2}\Big]_{t=0}$$

$$= p\Big[6e^{-3t}(e^{-t}-q)^{-4} - 6e^{-2t}(e^{-t}-q)^{-3} +$$
$$e^{-t}(e^{-t}-q)^{-2}\Big]_{t=0}$$

$$= p\Big[6(1-q)^{-4} - 6(1-q)^{-3} + (1-q)^{-2}\Big]$$

$$= p\left\{\frac{6}{p^4} - \frac{6}{p^3} + \frac{1}{p^2}\right\}$$

$$= p\left\{\frac{6-6p+p^2}{p^4}\right\} = \frac{6-6p+p^2}{p^3}$$

$$\mu_4' = p\frac{d}{dt}\Big[6e^{-3t}(e^{-t}-q)^{-4} - 6e^{-2t}(e^{-t}-q)^{-3}$$
$$+ e^{-t}(e^{-t}-q)^{-2}\Big]_{t=0}$$

$$= p\Big[6\{4e^{-3t}(e^{-t}-q)^{-5}e^{-t}\} + 6\{-3e^{-3t}(e^{-t}-q)^{-4}\}$$
$$-6\{3e^{-2t}(e^{-t}-q)^{-4}e^{-t}\} - 6\{-2e^{-2t}(e^{-t}-q)^{-3}\}$$
$$+\{2e^{-t}(e^{-t}-q)^{-3}e^{-t} - e^{-t}(e^{-t}-q)^{-2}\}\Big]_{t=0}$$

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$$= p\left[24(1-q)^{-5} - 18(1-q)^{-4} - 18(1-q)^{-4} + 12(1-q)^{-3} + 2(1-q)^{-3} - (1-q)^{-2}\right]$$

$$= p\left[\frac{24}{p^5} - \frac{36}{p^4} + \frac{14}{p^3} - \frac{1}{p^2}\right]$$

$$= \frac{p\{24 - 36p + 14p^2 - p^3\}}{p^5}$$

$$\Rightarrow \mu_4' = \frac{24 - 36p + 14p^2 - p^3}{p^4}$$

M.G.F. About Mean.

$$M_\mu(t) = E(e^{t(x-\mu)})$$

where μ is the mean

$$\Rightarrow M_\mu(t) = \sum_{x=1}^{\infty} e^{t(x-\mu)} pq^{x-1}$$

$$= e^{-\mu t} \sum_{x=1}^{\infty} e^{tx} pq^{x-1}$$

$$= e^{-\mu t} p(e^{-t} - q)^{-1}$$

$$= e^{-\frac{1}{p}t} p(e^{-t} - q)^{-1}$$

$$= p\left[e^{\frac{1}{p}t}(e^{-t} - q)\right]^{-1}$$

$$= p\left[e^{(\frac{1}{p}-1)t} - qe^{1/p\, t}\right]^{-1}$$

$$= p\left[e^{\frac{q}{p}t} - qe^{1/p\, t}\right]^{-1}$$

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$$\boxed{M_\mu(t) = p\left(e^{q/p\, t} - q e^{1/p\, t}\right)^{-1}}$$

$$\mu_r = \frac{d^r}{dt^r} p\left(e^{q/p\, t} - q e^{1/p\, t}\right)^{-1}\Big|_{t=0}$$

Solving similarly as negative binomial dis. we get

$\mu_1 = 0, \quad \mu_2 = q/p^2$

$\mu_3 = \frac{q}{p^3}(1+q)$

and $\mu_4 = \frac{q}{p^4}\left[p^2 + 9q\right]$.

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(1+q)^2}{q}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{p^2 + 9q}{q}.$$

Note:

Similarly, we can find cumulating generating function as -ve binomial distribution.

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Application of Binomial Distribu

Question (1)

A baseball player's batting average is 0.25. what is the probab that he gets exactly 1 hit in his next 5 times at bat.

Solution:

Since it is binomial distributio with $n = 5$ and $p = 0.25$

$$q = 1 - p = 0.75$$

let X denotes the no. of hits

$$P(X=1) = {}^5C_1 (0.25)^1 (0.75)^4$$

$$= \boxed{0.3955}$$

Question (2)

A multiple choice quiz has 15 questions, each with four possible answers of which only 1 is correct. what is the probability that sheer guesswork yields from 5 to 10 correct answer.

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Solution

Since it is binomial distribution with $n = 15$, $p = \frac{1}{4}$, $q = 1 - p = \frac{3}{4}$

let X denotes the no. of correct answe

we have to find $P(X = x) = P(5 \le x \le 10)$

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$P(5 \le x \le 10) = P(X=5) + P(X=6) + P(X=7)$
$+ P(X=8) + P(X=9) + P(X=10)$

$= {}^{15}C_5 \left(\frac{1}{4}\right)^5 \left(\frac{3}{4}\right)^{10} + {}^{15}C_6 \left(\frac{1}{4}\right)^6 \left(\frac{3}{4}\right)^9 + {}^{15}C_7 \left(\frac{1}{4}\right)^7 \left(\frac{3}{4}\right)^8$

$+ {}^{15}C_8 \left(\frac{1}{4}\right)^8 \left(\frac{3}{4}\right)^7 + {}^{15}C_9 \left(\frac{1}{4}\right)^9 \left(\frac{3}{4}\right)^6 + {}^{15}C_{10} \left(\frac{1}{4}\right)^{10} \left(\frac{3}{4}\right)^5$

$= 0.1651 + 0.0917 + 0.0393 + 0.0131 + 0.0034$
$+ 0.0007$

$= \boxed{0.3133}$

Question (3)

The Probability that a patient recovers from a delicate heart operation is 0.9. what is the probability that exactly 5 of the next 7 patients having this operation survive.

Solution:-

Since it is a binomial distribution with $n = 7$, $p = 0.90$, $q = 0.10$

let x denotes the no. of patients recovered from delicate operation.

Then $P(X=5) = {}^7C_5 (0.90)^5 (0.10)^2$

$P(X=5) = 0.1240$

$= \boxed{0.1240}$

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Question (4)

One fourth of the female freshmen entering a virginia college are out-of-state students. If the students are assigned at random to the dormitories, 3 to a room, what is the probability that in one room almost 2 of the 3 are out of state

Solution:

Since it is a binomial dist. with $n = 3$, $p = \frac{1}{4}$, $q = \frac{3}{4}$

let X denotes the no. of students which are out-of-state.

we have to calculate $P(X \leq 2)$

$$P(X = x) = {}^{n}C_{x}\, p^{x} q^{n-x}$$

$$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$$

$$= {}^{3}C_{0}\left(\frac{1}{4}\right)^{0}\left(\frac{3}{4}\right)^{3} + {}^{3}C_{1}\left(\frac{1}{4}\right)^{1}\left(\frac{3}{4}\right)^{2} + {}^{3}C_{2}\left(\frac{1}{4}\right)^{2}\left(\frac{3}{4}\right)^{1}$$

$$= 0.4219 + 0.4219 + 0.1406$$

$$= \boxed{0.9844}$$

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$$P(X \leq 2) = 1 - P(X > 2)$$

$$= 1 - P(X = 3)$$

$$= 1 - \left[{}^{3}C_{3}\left(\frac{1}{4}\right)^{3}\left(\frac{3}{4}\right)^{0}\right] = \boxed{0.9844}$$

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Question (5)

A Survey of the residents in a United States city showed that 20% performed a white telephone over any other colour available: what is Probability that more than one half of the next twenty telephones installed in this city will be white.

Solution:-

Since it is binomial dist. with $n = 20$, $p = 0.20$, $q = 0.80$, let X denotes the no. of white telephones:

we have to find $P(X > 10)$

$$P(X > 10) = P(X=11) + P(X=12) + P(X=13) + P(X=14) + P(X=15) + P(X=16) + P(X=17) + P(X=18) + P(X=19) + P(X=20)$$

$$P(X > 10) = {}^{20}C_{11}(0.20)^{11}(0.80)^{9} + {}^{20}C_{12}(0.20)^{12}(0.80)^{8} + {}^{20}C_{13}(0.20)^{13}(0.80)^{7} + {}^{20}C_{14}(0.20)^{14}(0.80)^{6} + {}^{20}C_{15}(0.20)^{15}(0.80)^{5} + {}^{20}C_{16}(0.20)^{16}(0.80)^{4} + {}^{20}C_{17}(0.20)^{17}(0.80)^{3} + {}^{20}C_{18}(0.20)^{18}(0.80)^{2} + {}^{20}C_{19}(0.20)^{19}(0.80)^{1} + {}^{20}C_{20}(0.20)^{20}(0.80)^{0}$$

$$= 0.00046 + 0.00008657 + 0.00001332 + 0.000001664 + 0.000000166 + 0.000000013 + 0.00000007 + 0.00000003 + 0.0000008 + 0.00000001 = \boxed{0.000563414}$$

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Question (6)

Assuming that half of the population are consumers of choclate, so that the chance of an individual being a consumer is $\frac{1}{2}$, and assuming that 100 investigators each take 10 individuals to see whether they are consumers. How many investigators would you expect to report that 3 people or less were consumer.

Solution

Since it is binomial distribution with $n=10$, $p=0.5$, $q=0.5$ and $N=100$

let x denotes the no. of consumers

we have to calculate $N[P(X \leq 3)] = ?$

$$P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= {}^{10}C_0 (0.5)^0 (0.5)^{10} + {}^{10}C_1 (0.5)^1 (0.5)^9 + {}^{10}C_2 (0.5)^2 (0.5)^8 + {}^{10}C_3 (0.5)^3 (0.5)^7$$

$$= 0.0009766 + 0.009766 + 0.04395 + 0.117187$$

$$= \boxed{0.171875}$$

Now $N[P(X \leq 3)] = N[0.171875]$

$$= 100[0.171875]$$

$$= 17.19 \approx 17 \text{ Persons}$$

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Question (7)

Find Probability of obtaining exactly three 2's if an ordinary die is tossed 5 times.

Solution:-

Since it is a binomial distribution with $n=5$, $p=\frac{1}{6}$, $q=\frac{5}{6}$

let X denotes the no. of 2's

Then we calculate $P(X=3)$

$$P(X=3) = {}^5C_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2$$

$$= \boxed{0.0321}$$

Question (8)

Find Probability of getting five heads and seven tails in 12 flips of a balanced coin.

Solution:-

Since it is a binomial distribution with $n=12$, $p=\frac{1}{2}=q$

let X denotes the no. of heads appear

we have to calculat $P(X=5)$

$$P(X=5) = {}^{12}C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^7$$

$$= 0.1934$$

$$P(X=5) = \boxed{0.1934}$$

Answer

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Question (9)

A multiple-choice test consists of eight questions and three answers to each question (of which only one is correct). If a student answers each question by rolling a balanced die and checking the first, 2nd and third answer if he gets 1 or 2, 3 or 4, 5 or 6 respectively. What is the probability that he will get exactly four correct answers.

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Solution:-

Since it is binomial distribution with $n = 8$, $p = \frac{1}{3}$, $q = \frac{2}{3}$

Let X denotes the no. of correct answers.

We have to calculate $P(X=4)$

$$P(X=4) = {}^8C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^4$$

$$= \boxed{0.1707}$$

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Question (10) /318

An irregular six faced die is thrown and the expectation that in 10 throws it will give five even numbers is twice the expectation that it will give four even numbers. How many times 10000 sets of ten throws would you expect it to give no even numbers.

Solution:-

Since it is binomial dist. with $n = 10$, $N = 10000$

Then the expected number of getting 5 even no in 10 throws

$$N\,{}^{10}C_5\, p^5 q^5 \quad \text{———} \quad (i)$$

and the expected no. of getting 4 even no. in 10 throws

$$N\,{}^{10}C_4\, p^4 q^6 \quad \text{———} \quad (ii)$$

According to given condition

$$N\,{}^{10}C_5\, p^5 q^5 = 2N\,{}^{10}C_4\, p^4 q^6$$

$$(252)\,N\, p^5 q^5 = (420)\,N\, p^4 q^6$$

$$\frac{252\,\cancel{N}\, p^5 q^5}{420\,\cancel{N}\, p^4 q^6} = 1$$

$$\frac{0.6\,p}{q} = 1$$

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$(0.6)P = q$

$(0.6)P = 1-P$

$0.6P + P = 1$

$1.6P = 1$

$\boxed{P = 0.625}$

and $q = 1-P = 1-0.625$

$\boxed{q = 0.375}$

Let X denotes the no. of even no.

$P(X=0) = {}^{10}C_0 (0.625)^0 (0.375)^{10}$

$= 0.00005499$

Now expectation that it will give 0 even numbers

$NP(X=0) = 10000(0.00005499)$

$0.5499 \approx 1$ even no

Question (11)

A random variable is binomially distributed with mean 3 and variance 2.

Find $P(X=7)$

Solution:-

Given data mean = 3 and variance = 2

Since the distribution is binomial. So

mean $= nP = 3$ —— (i)

variance $= nPq = 2$ —— (ii)

Dividing eq (i) and (ii)

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$$\frac{npq}{np} = \frac{2}{3} \Rightarrow q = \frac{2}{3}$$

So $p = 1-q = \frac{1}{3}$, $n=9$

Now we calculate $P(X=7)$

$$P(X=7) = {}^9C_7 \left(\frac{1}{3}\right)^7 \left(\frac{2}{3}\right)^2$$

$$P(X=7) = 0.0073$$

Question (12)

In $b(x, n, p)$; $\beta_1 = \frac{1}{15}$, $\beta_2 = \frac{89}{30}$ were determined when 100 batches of seeds were allowed to germinate on damp filter paper in a laboratory.

Determine the binomial distribution and calculate frequency for $x=8$ where $p>q$

Solution:-

Given for binomial distribution

$\beta_1 = \frac{1}{15}$, $\beta_2 = \frac{89}{30}$, $N=100$, $x=8$, $p>q$

$n=?$, $p=?$, $q=?$

we know that

$$\beta_1 = \frac{(\mu_3)^2}{(\mu_2)^3} \quad \text{and} \quad \beta_2 = \frac{\mu_4}{(\mu_2)^2}$$

in binomial distribution

$\mu_2 = npq$, $\mu_3 = npq(q-p)$, $\mu_4 = 3n^2p^2q^2 + npq(1-6pq)$

Putting values in β_1 and β_2

$$\beta_1 = \frac{n^2p^2q^2(q-p)^2}{n^3p^3q^3} = \frac{(q-p)^2}{npq} = \frac{1}{15} \quad \text{——— (i)}$$

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$$\beta_2 = \frac{3n^2p^2q^2 + npq(1-6pq)}{n^2p^2q^2} = \frac{3n^2p^2q^2}{n^2p^2q^2} + \frac{npq(1-6pq)}{n^2p^2q^2}$$

$$= 3 + \frac{(1-6pq)}{npq} = \frac{89}{30}$$

$$\frac{1-6pq}{npq} = \frac{89}{30} - 3 = \frac{-1}{30} \quad \text{——— (2)}$$

Dividing eq (1) and (2), we have

$$\frac{\frac{1-6pq}{\cancel{npq}}}{\frac{(q-p)^2}{\cancel{npq}}} = \frac{-1/30}{1/15}$$

$$= \frac{1-6pq}{(q-p)^2} = \frac{-1}{30} \times \frac{15}{1} = \frac{-1}{2}$$

$$1-6pq = \frac{-1}{2}(q-p)^2$$

$$-2\left((1-6pq)\right) = (q-p)^2$$

$$-2\left(1-6p(1-p)\right) = (1-p-p)^2$$

$$-2 + 12p - 12p^2 = 1 + 4p^2 - 4p$$

$$4p^2 + 12p^2 - 4p - 12p + 1 + 2 = 0$$

$$16p^2 - 16p + 3 = 0$$

$$16p^2 - 12p - 4p + 3 = 0$$

$$4p(4p-3) - 1(4p-3) = 0$$

$$(4p-1)(4p-3) = 0$$

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$P = \frac{1}{4}$, $P = \frac{3}{4}$

Now for q

$q = 1 - P$

when $P = \frac{1}{4} \Rightarrow q = 1 - \frac{1}{4} = \frac{3}{4}$

when $P = \frac{3}{4} \Rightarrow q = 1 - \frac{3}{4} = \frac{1}{4}$

we have two possibilities

$P = \frac{1}{4}$, $q = \frac{3}{4}$ (OR) $P = \frac{3}{4}$, $q = \frac{1}{4}$

But according to condition $P > q$

Hence $P = \frac{3}{4}$, $q = \frac{1}{4}$

Put in eq (1)

$$\frac{\left(\frac{1}{4} - \frac{3}{4}\right)^2}{n\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)} = \frac{1}{15}$$

$$\frac{(1/2)^2}{n} = \frac{1}{15}\left(\frac{3}{16}\right)$$

$$\frac{1}{4n} = \frac{1}{80}$$

$$4n = 80 \Rightarrow n = \frac{80}{4} = 20$$

$$\boxed{n = 20}$$

Hence the desired binomial frequency distribution for $x = 8$ is

$NP(X=8) = N\,{}^nC_x\, P^x q^{n-x}$

$= 100\,{}^{20}C_8 \left(\frac{3}{4}\right)^8 \left(\frac{1}{4}\right)^{12}$

$NP(X=8) = \boxed{0.0751}$

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Now we calculate complete binomial dist

x	$P(X=x) = {}^nC_x p^x q^{n-x}$
0	${}^{20}C_0 (\frac{3}{4})^0 (\frac{1}{4})^{20}$ = 0.000000909
1	${}^{20}C_1 (\frac{3}{4})^1 (\frac{1}{4})^{19}$ = 0.000000054
2	${}^{20}C_2 (\frac{3}{4})^2 (\frac{1}{4})^{18}$ = 0.000000001
3	${}^{20}C_3 (\frac{3}{4})^3 (\frac{1}{4})^{17}$ = 0.000000027
4	${}^{20}C_4 (\frac{3}{4})^4 (\frac{1}{4})^{16}$ = 0.000000356
5	${}^{20}C_5 (\frac{3}{4})^5 (\frac{1}{4})^{15}$ = 0.000003426
6	${}^{20}C_6 (\frac{3}{4})^6 (\frac{1}{4})^{14}$ = 0.000025570
7	${}^{20}C_7 (\frac{3}{4})^7 (\frac{1}{4})^{13}$ = 0.000154192
8	${}^{20}C_8 (\frac{3}{4})^8 (\frac{1}{4})^{12}$ = 0.000751687
9	${}^{20}C_9 (\frac{3}{4})^9 (\frac{1}{4})^{11}$ = 0.00300675
10	${}^{20}C_{10} (\frac{3}{4})^{10} (\frac{1}{4})^{10}$ = 0.009922275
11	${}^{20}C_{11} (\frac{3}{4})^{11} (\frac{1}{4})^{9}$ = 0.02706075
12	${}^{20}C_{12} (\frac{3}{4})^{12} (\frac{1}{4})^{8}$ = 0.060886689
13	${}^{20}C_{13} (\frac{3}{4})^{13} (\frac{1}{4})^{7}$ = 0.112406195
14	${}^{20}C_{14} (\frac{3}{4})^{14} (\frac{1}{4})^{6}$ = 0.168609293
15	${}^{20}C_{15} (\frac{3}{4})^{15} (\frac{1}{4})^{5}$ = 0.202331151
16	${}^{20}C_{16} (\frac{3}{4})^{16} (\frac{1}{4})^{4}$ = 0.189685454
17	${}^{20}C_{17} (\frac{3}{4})^{17} (\frac{1}{4})^{3}$ = 0.133895615
18	${}^{20}C_{18} (\frac{3}{4})^{18} (\frac{1}{4})^{2}$ = 0.066947807
19	${}^{20}C_{19} (\frac{3}{4})^{19} (\frac{1}{4})^{1}$ = 0.021141412
20	${}^{20}C_{20} (\frac{3}{4})^{20} (\frac{1}{4})^{0}$ = 0.003171211

Sum = 1.000000000

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Question (13) (324)

Let $M_x(t) = \left(\frac{1}{4} + \frac{3}{4}e^t\right)^{12}$ Find

(i) $E(\ )$, (ii) $V(X)$, (iii) $P(X \geq 10)$

Solution:-

Since in binomial distribution

$M_x(t) = (q + pe^t)^n$ compare it

with $M_x(t) = \left(\frac{1}{4} + \frac{3}{4}e^t\right)^{12}$

we get $q = \frac{1}{4}$, $p = \frac{3}{4}$, $n = 12$

(i) $E(X) = ?$

Since $E(X) = \text{mean} = np$

$= 12\left(\frac{3}{4}\right) = 9$

$\boxed{E(X) = 9}$

(ii) $V(X) = ?$

Since variance $= npq$

$= 12\left(\frac{3}{4}\right)\left(\frac{1}{4}\right) = \frac{9}{4}$

$\boxed{V(X) = 9/4}$

(iii) $P(X \geq 10) = ?$

$P(X \geq 10) = P(X = 10) + P(X = 11) + P(X = 12)$

$= {}^{12}C_{10}\left(\frac{3}{4}\right)^{10}\left(\frac{1}{4}\right)^2 + {}^{12}C_{11}\left(\frac{3}{4}\right)^{11}\left(\frac{1}{4}\right)^1 + {}^{12}C_{12}\left(\frac{3}{4}\right)^{12}\left(\frac{1}{4}\right)^0$

$= 0.2323 + 0.1267 + 0.03168$

$P(X \geq 10) = \boxed{0.39068}$

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Question (14) (3223)

Fit a binomial distribution to the following data

x	0	1	2	3	4
f	30	62	46	10	2

Solution:-

x	f	xf
0	30	0
1	62	62
2	46	92
3	10	30
4	2	8
	150	192

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Since in binomial dist. mean $= \bar{x} = np$

So $np = \text{mean} = \bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{192}{150} = 1.28$

Here $\boxed{n=4}$

So, $np = 1.28$

$\Rightarrow 4p = 1.28 \Rightarrow \boxed{p = 0.32}, \boxed{q = 0.68}$

Hence expected frequencies are

x	$P(x) = {}^nC_x p^x q^{n-x}$	$150P(x)$
0	${}^4C_0 (0.32)^0 (0.68)^4 = 0.2138$	32
1	${}^4C_1 (0.32)^1 (0.68)^3 = 0.4025$	60
2	${}^4C_2 (0.32)^2 (0.68)^2 = 0.2841$	43
3	${}^4C_3 (0.32)^3 (0.68)^1 = 0.08913$	13
4	${}^4C_4 (0.32)^4 (0.68)^0 = 0.01049$	2

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Question (15)

Let $\left(\frac{3}{4}+\frac{1}{4}\right)^{10}$ represents $b(x, n, p)$

Find mean and Variance

Solution:-

Since in binomial distribution

$b(x, n, p) = (q+p)^n$ compare it

with $\left(\frac{3}{4}+\frac{1}{4}\right)^{10}$ we get

$p = \frac{1}{4}$, $q = \frac{3}{4}$ and $n = 10$

Now we find mean and Variance

Since mean $= np = 10\left(\frac{1}{4}\right) = \boxed{2.5}$

and Variance $= npq = 10\left(\frac{1}{4}\right)\left(\frac{3}{4}\right) = \boxed{1.875}$

Hypergeometric Distribution

Question (16)

If five cards are dealt from a standard deck of 52 cards, what is the probability that 3 will be heart?

Solution:-

Given that $n = 5$, $N = 52$, $k = 13$

Let X denotes the no. of heart cards

$$P(X=x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$$

$$P(X=3) = \frac{\binom{13}{3}\binom{52-13}{5-3}}{\binom{52}{5}} = \frac{\binom{13}{3}\binom{39}{2}}{\binom{52}{5}}$$

$$P(X=3) = \boxed{0.082}$$

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Question (17) (327)

The names of 5 men and 5 women are written on slip of paper and placed in a hat. Four names are drawn. what is the Probability that 2 are men and 2 are women.

Solution:-

Given $N = 5+5 = 10$, $n = 4$, $k = 5$

let x denotes the no. of men

$$P(X=x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$$

$$P(X=2) = \frac{\binom{5}{2}\binom{10-5}{4-2}}{\binom{10}{4}}$$

$$= \frac{\binom{5}{2}\binom{5}{2}}{\binom{10}{4}} = \frac{10(10)}{210} = 0.4762$$

$$P(X=2) = \boxed{0.4762}$$

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Question (18)

what is the Probability that a poker hand will contain exactly 2 aces

Solution:-

Total cards : $N = 52$

Poker hand : $n = 5$

Ace cards : $k = 4$

let x denotes the no. of aces

$$P(X=x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{}}$$

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$$P(X=2) = \frac{\binom{4}{2}\binom{52-4}{5-2}}{\binom{52}{5}}$$

$$= \frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}} = \frac{6(17296)}{2598960}$$

$$P(X=2) = \boxed{0.040}$$

Question (19)

A homeowner plants 6 bulbs selected at random from a box containing 4 tulip bulbs and 4 daffodil bulbs. What is the probability that he planted 2 daffodil and 4 tulip bulbs?

Solution:-

Here we have

$N = 8, \quad n = 6, \quad \cancel{r}_2 = 4$

Let X denotes the no. of daffodil bulb

$$P(X=x) = \frac{\binom{\cancel{r}_2}{x}\binom{N-\cancel{r}_2}{n-x}}{\binom{N}{n}}$$

$$P(X=2) = \frac{\binom{4}{2}\binom{8-4}{6-2}}{\binom{8}{6}}$$

$$= \frac{\binom{4}{2}\binom{4}{4}}{\binom{8}{6}} = 0.2143$$

$$P(X=2) = \boxed{0.2143}$$

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Question (20): (321)

To avoid detection at customs, a traveller has placed six narcotic tablets containing nine vitamin pills that are similar in appearance. if the customs official selects 3 of the tablets at random for analysis, what is the probability that the traveller will be arrested for illegal possession of narcotics?

Solution:-

we have $N=15$, $n=3$, $k=6$

let X denotes the no. of narcotic tablets. The traveller will be arrested having atleast 1 pill so $x = 1, 2, 3$

So $P(1 \le X \le 3) = ?$

$$P(X=x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$$

$$P(1 \le X \le 3) = P(X=1) + P(X=2) + P(X=3)$$

$$= \frac{\binom{6}{1}\binom{15-6}{3-1}}{\binom{15}{3}} + \frac{\binom{6}{2}\binom{15-6}{3-2}}{\binom{15}{3}} + \frac{\binom{6}{3}\binom{15-6}{3-3}}{\binom{15}{3}}$$

$$= \frac{\binom{6}{1}\binom{7}{2}}{\binom{15}{3}} + \frac{\binom{6}{2}\binom{7}{1}}{\binom{15}{3}} + \frac{\binom{6}{3}\binom{7}{0}}{\binom{15}{3}} = \boxed{0.82}$$

$P(1 \le X \le 3) = \boxed{0.82}$

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Question (21)

(330)

What is the probability that a waitress will refuse the serve alcoholic beverages to only 2 minors if she randomly checks the I.D's of 6 students from among 9 students of which 4 are not of legal age?

Solution:-

Here $N=9$, $n=6$, $k=4$, $x=2$

Let X denotes the no. of minors

$$P(X=x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$$

$$P(X=2) = \frac{\binom{4}{2}\binom{9-4}{6-2}}{\binom{9}{6}} = \frac{\binom{4}{2}\binom{5}{4}}{\binom{9}{6}} = \frac{6(5)}{84}$$

$$P(X=2) = \boxed{0.3571}$$

Question (22)

Lots of 40 components each are called acceptable if they contain no more than 3 defectives. The procedure for sampling the lot is to select 5 components at random and to reject the lot if a defectives found. What is the probability that exactly 1 defective is found if there are 3 defectives in the entire lot.

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Solution:-

Here $N = 40$, $n = 5$, $k = 3$ and $x = 1$

Let X denotes the no. of defective compo.

$$P(X=x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$$

$$P(X=1) = \frac{\binom{3}{1}\binom{40-3}{5-1}}{\binom{40}{5}} = \frac{\binom{3}{1}\binom{37}{4}}{\binom{40}{5}} = \frac{(3)(66045)}{658008}$$

$$P(X=1) = \boxed{0.3011}$$

Question (23)

As part of an air-pollution servey, an inspector decides to examine the exhaust of six of a company's 24 trucks. If four of company's trucks emit excessive amounts of pallutants, what is the probability that non of them will be included in the inspectors sample

Solution:-

Here $N = 24$, $k = 6$, $n = 4$, $x = 0$

$$P(X=x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$$

$$P(X=0) = \frac{\binom{6}{0}\binom{24-6}{4-0}}{\binom{24}{4}} = \frac{\binom{6}{0}\binom{18}{4}}{\binom{24}{4}} = \frac{3060}{10626}$$

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Question(27) (5.36)

Find the Probability of being dealt a bridge hand containing 5 spades, 2 hearts, 3 diamonds and 3 clubs.

Solution:-

No. of cards of spades = 13

No. of cards of hearts = 13

No. of cards of diamonds = 13

No. of cards of clubs = 13

Total cards = 52

Let X_1, X_2, X_3, X_4 denote the spade, hearts, diamonds and clubs cards.

$P_1 = P(X_1) = \frac{13}{52}$

$P_2 = P(X_2) = 13/52$

$P_3 = P(X_3) = 13/52$

$P_4 = P(X_4) = 13/52$

Hence

$$P(X_1, X_2, X_3, X_4) = \frac{13!}{5!\,2!\,3!\,3!}\left(\frac{13}{52}\right)^5\left(\frac{13}{52}\right)^2\left(\frac{13}{52}\right)^3\left(\frac{13}{52}\right)^3$$

$$= \frac{6227020800}{(120)(2)(6)(6)}(0.000000014)$$

$$= \boxed{0.01074}$$

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Question (28) (337)

The painted light bulbs produced by a company are 50% red, 30% blu and 20% green. In a sample of 5 bulbs, Find the Probability that 2 are red, one is green and 2 bl

Solution:-

Let X_1, X_2, X_3 denote the red, green and blue bulbs

$P_1 = P(X_1) = 50\% = 0.50$

$P_2 = P(X_2) = 30\% = 0.30$

$P_3 = P(X_3) = 20\% = 0.20$

Hence,

$$P(X_1, X_2, X_3) = \frac{5!}{2!\,1!\,2!}(0.50)^2(0.30)^1(0.20)^2$$

$$= \frac{120}{4}(0.003) = 0.09$$

$$P(X_1, X_2, X_3) = \boxed{0.09}$$

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Question (29)

The Probabilities are 0.4, 0.2, 0.3, 0.1 respectively that a delegate to a certain convention arrived by air, bus, automobile and train. what is the Probability that among 9 delegates randomly selected at this convention, 3 arrived by air, 3 by bus 1 by automobile and 2 arrived by train

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Solution:-

Let X_1, X_2, X_3, X_4 denote that a delegate arrived by air, bus, automobile and train respectively

$P_1 = P(X_1) = 0.4$

$P_2 = P(X_2) = 0.2$

$P_3 = P(X_3) = 0.3$

$P_4 = P(X_4) = 0.1$

Hence

$$P(X_1, X_2, \ldots, X_4) = \frac{9!}{3!\,3!\,1!\,2!}(0.4)^3(0.2)^3(0.3)^1(0.1)^2$$

$$\frac{362880}{(6)(6)(2)}(0.000001536)$$

$P(X_1, X_2, X_3, X_4) = \boxed{0.0077}$

Question (30)

According to a genetics theory, a certain cross of guinea pigs will result in red, black, and white offspring in the ratio 8:4:4. Find probability that among 8, offspring 5 will be red, 2 black and one white.

Solution:-

No. of red pigs = 8

No. of black pigs = 4

No. of white off spring = 4

Total pigs = 16

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Let X_1, X_2, X_3 denote the red, black and white offspring pigs. Then

$P_1 = P(X_1) = 8/16$

$P_2 = P(X_2) = 4/16$

$P_3 = P(X_3) = 4/16$

Hence,

$$P(X_1, X_2, X_3) = \frac{8!}{5!\,2!\,1!}\left(\frac{8}{16}\right)^5\left(\frac{4}{16}\right)^2\left(\frac{4}{16}\right)^1$$

$$= \frac{40320}{(120)(2)}(0.0004883)$$

$$P(X_1, X_2, X_3) = \boxed{0.082}$$

Question (31)

A certain city has three television stations. During prime time on saturday nights, channel 12 has 50% of the viewing audience, channel 10 has 30% of the viewing audience, channel 3 has 20% of the viewing audience.

Find the probability that among 8 television viewers in that city, randomly chosen on saturday night, five will be watching channel 12, two will be watching 10 and one watching channel 3.

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Solution:-

Let X_1, X_2, X_3 denote the channel no. 12, 10 and channel 3 respectively.

Then $P_1 = P(X_1) = 50\% = 0.5$

$P_2 = P(X_2) = 30\% = 0.3$

$P_3 = P(X_3) = 20\% = 0.2$

Hence

$$P(X_1, X_2, X_3) = \frac{8!}{5!\,2!\,1!}(0.5)^5(0.3)^2(0.2)^1$$

$$= \frac{40320}{120(2)}(0.0005625)$$

$$P(X_1, X_2, X_3) = \boxed{0.0945}$$

Question (32)

9 of 18 defective glass bricks include 10 that have cracks but no discoloration, five that have discoloration but no cracks, and three that have cracks and discoloration.

What is the probability that among six of the bricks, three will have cracks but no discoloration, one will have discoloration but no cracks, and two will have cracks and discoloration?

Solution:-

No. of glasses that have cracks but no discoloration = 10

No. of glasses that have discoloration but no cracks = 5

No. of glasses that have cracks and discoloration = 3

Total no. of glasses = 18

Let X_2, X_1, X_3 denotes the type of defective glasses having discoloration but no cracks, having cracks but no discoloration and having both cracks and discoloration. Then

$P_1 = P(X_1) = 10/18$

$P_2 = P(X_2) = 5/18$

$P_3 = P(X_3) = 3/18$

Hence

$$P(X_1, X_2, X_3) = \frac{6!}{3!\,1!\,2!}\left(\frac{10}{18}\right)^3\left(\frac{5}{18}\right)^1\left(\frac{3}{18}\right)^2$$

$$= \frac{720}{6(2)}(0.001323)$$

$P(X_1, X_2, X_3) = \boxed{0.07938}$

Question (33)

According to USA, 4 million workers in the general workforce, 5.8% tested positive for drugs. of those testing positive 22.5% were cocaine users and 54.4% marijuana users. what is the probability

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that of 10 workers testing positive, 2 are cocaine users, 5 marijuana and 3 users of other drugs.

Solution:-

No. of Persons tested positive drugs

5.8% of 4 million

$$\frac{5.8}{100} \times 4000000 = \boxed{232000}$$

No. of cocaine users = 22.5% of 232000

$$= \frac{22.5}{100} \times 232000 = \boxed{52200}$$

No. of marijuana users = 54.4% of 232000

$$= \frac{54.4}{100} \times 232000 = \boxed{126208}$$

No. of other users = 232000 − 52200 − 126208

$$= \boxed{53592}$$

Let X_1, X_2, X_3 denotes the cocaine users, marijuana users and other drug users

Then

$$P_1 = P(X_1) = \frac{52200}{232000} = 0.225$$

$$P_2 = P(X_2) = \frac{126208}{232000} = 0.544$$

$$P_3 = P(X_3) = \frac{53592}{232000} = 0.231$$

Then

$$P(X_1, X_2, X_3) = \frac{10!}{2!\,5!\,3!}(0.225)^2(0.544)^5(0.231)^3$$

$$= \frac{3628800}{2(120)(6)}(0.00002973) = \boxed{0.075}$$

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Question (34) (343)

If a pair of dice is tossed 6 times what is the Probability of obtaining a total of 7 or 11 twice, a matching pair once, and any other combination 3 times?

Solution:-

we a pair of dice is rolled the total no. of Sample Points 36

let X_1, X_2, X_3 denote the total of 7 or 11, matching pair and any other Combination respectively

$$P_1 = P(X_1) = \frac{6}{36} + \frac{2}{36} = \frac{8}{36}$$

$$P_2 = P(X_2) = \frac{6}{36}$$

$$P_3 = P(X_3) = \frac{22}{36}$$

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Hence

$$P(X_1, X_2, X_3) = \frac{6!}{2!!!\,3!} \left(\frac{8}{36}\right)^2 \left(\frac{6}{36}\right)^1 \left(\frac{22}{36}\right)^3$$

$$= \frac{720}{2(6)} (0.001878)$$

$$P(X_1, X_2, X_3) = \boxed{0.1127}$$

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Negative binomial + Geometric Dist

Negative binomial P.d is

$$b^*(x; k, p) = \binom{x-1}{k-1} p^k q^{x-k} \quad ; x = k, k+1, \ldots$$

In case of geometric Dist $k = 1$

$$g(x, p) = \binom{x-1}{0} p q^{x-1} = p q^{x-1}$$

Question (35)

If the probability is 0.40 that a child exposed to a certain categories disease will catch it, what is the Probability that the tenth child exposed to the disease will be the third to catch it.

Solution:

Here $p = 0.40$, $x = 10$, $k = 3$, $q = 0.60$

let X denotes the no. of child

$$b^*(x; k, p) = \binom{x-1}{k-1} p^k q^{x-k}$$

$$b^*(10; 3, 0.4) = \binom{10-1}{3-1} (0.40)^3 (0.60)^{10-3}$$

$$= \binom{9}{2} (0.4)^3 (0.6)^7$$

$$b^*(10; 3, 0.4) = \boxed{0.0645}$$

Question (36)

If a hunter is shooting at a target, what is the Probability that 5th fire is the 2nd hit; given the Probability of hitting the target is 0.20

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(345)

Solution:-

Here $p = 0.2$, $q = 0.8$, $x = 5$, $k = 2$

$b^*(x; k, p) = \binom{x-1}{k-1} p^k q^{x-k}$

$b^*(5; 2, 0.2) = \binom{5-1}{2-1} (0.2)^2 (0.8)^{5-2}$

$= \binom{4}{1} (0.2)^2 (0.8)^3$

$= \boxed{0.08192}$

Question (37)

An item is produced in large number The machine is known to produce 5% defectives. A quality control inspector is examing the items by taking them a random. what is the probability that atleast 4 items are to be examined in order to get 2 defectives:

Solution:-

Here $p = 0.05$, $q = 0.95$, $k = 2$

Let x denote the no. of items to be exam

we calculate $P(x \geq 4) = ?$

$P(x \geq 4) = 1 - \sum_{x=2}^{3} b^*(x; k, p)$

$= 1 - \left[b^*(2; 2, 0.05) + b^*(3; 2, 0.05) \right]$

$= 1 - \left[\binom{2-1}{2-1} (0.05)^2 (0.95)^0 + \binom{3-1}{2-1} (0.05)^2 (0.95)^{3-2} \right]$

$P(x \geq 4) = 1 - (0.0025 + 0.00475) = \boxed{0.99275}$

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Question (38) (38)

If the probability is 0.75 that an applicant for a drivers license will pass the road test on any given try, what is the probability that applicant will finally pass the 4th test first time.

Solution:-

Given that $x=4$, $P=0.75$, $q=0.25$

By using the Geometric distribution

$$P(X=x) = g(x,p) = pq^{x-1}$$

$$g(4;0.75) = (0.75)(0.25)^{4-1}$$

$$= (0.75)(0.25)^3$$

$$g(4,0.75) = \boxed{0.01172}$$

Question (39)

If the Probability that a person will believe a rumour about the retirement of a certain politician is 0.25. what is the Probability that

(a) the sixth person to hear the rumour will be first to believe it.

(b) the 12th person to hear the rumour will be the fourth to believe it.

Solution:

(a) Let X denotes the number of person who hears the rumour.

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Since it is geometric distribution

with $p = 0.25$, $x = 6$, $q = 0.75$

$$P(X=x) = g(x;p) = pq^{x-1}$$

$$P(X=6) = g(6;0.25) = (0.25)(0.75)^{6-1}$$

$$= (0.25)(0.75)^5$$

$$P(X=6) = \boxed{0.059}$$

(b) Let X denotes the no. of person who hears the rumour

Since it is a negative binomial dist

with $p = 0.25$, $q = 0.75$, $x = 12$, $k = 4$

$$b^*(x;k,p) = \binom{x-1}{k-1} p^k q^{x-k}$$

$$b^*(12;4,0.25) = \binom{12-1}{4-1}(0.25)^4(0.75)^{12-4}$$

$$= \binom{11}{3}(0.25)^4(0.75)^8$$

$$b^*(12;4,0.25) = \boxed{0.0645}$$

Question (40)

Find the probability that a person tossing 3 coins will get either all heads or all tails for the 2nd time on the 5th toss.

Solution:-

Given that $x = 5$, $k = 2$, $p = \frac{1}{4}$

By using Negative binomial

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$$b^*(x; k, p) = \binom{x-1}{k-1} p^k q^{x-k}$$

$$b^*(5; 2, \tfrac{1}{4}) = \binom{5-1}{2-1} \left(\tfrac{1}{4}\right)^2 \left(\tfrac{3}{4}\right)^{5-2}$$

$$= \binom{4}{1} \left(\tfrac{1}{4}\right)^2 \left(\tfrac{3}{4}\right)^3$$

$$b^*(5; 2, \tfrac{1}{4}) = \boxed{0.1055}$$

Question (41)

Find Probability that a person flipping a balanced coin requires 4 tosses to get a head

Solution:-

Since it is geometric dist with

$x = 4$, $p = \frac{1}{2} = q$

$$P(X = x) = g(x; p) = pq^{x-1}$$

$$P(X=4) = g(4; \tfrac{1}{2}) = \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)^{4-1}$$

$$= \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)^3 = \left(\tfrac{1}{2}\right)^4$$

$$g(4; \tfrac{1}{2}) = \frac{1}{16} = \boxed{0.0625}$$

Question (42)

Three people each toss a coin and the odd man pays for the cofee. If the coins all show heads or all show tails, they are tossed again. What is the Probability that a decision is reached in five tosses or fewer?

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Solution:-

To reach a decision on any trial, the coins must result in either 2 heads 1 tail or 2 tails and 1 head. The Probability of these events is computed by the binomial dist. because

each coin has two possible results, a head or a tail

The probability of getting a head is $p = \frac{1}{2}$ and remains same for each coin. three people toss the coins independent three coins are tossed in each case (i.e $n=3$). Therefore

$$P(2 \text{ heads } 1 \text{ tail}) = \binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) = \frac{3}{8}$$

$$P(1 \text{ head } 2 \text{ tails}) = \binom{3}{1}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2 = \frac{3}{8}$$

Hence the probability of reaching a decision is $\frac{3}{8} + \frac{3}{8} = \frac{6}{8} = \boxed{0.75}$

Next, we find the probability that a decision is reached in 5 tosses or fewer. we observe that

(i) we reach or do not reach a decision with each set,

(ii) The probability of reaching a decision is $p = 0.75$ and remains the same for all sets,

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(iii) the results of all sets are independent.
(iv) a variable no. of sets is required to produce 1 decision.

So we use negative binomial dist with $r = 1$, $p = 0.75$ and $x \leq 5$

Thus

$$P(X \leq 5) = \sum_{x=1}^{5} \binom{x-1}{r-1} (0.75)^{1} (0.25)^{x-r}$$

$$= \binom{0}{0}(0.75)(0.25)^{0} + \binom{2-1}{1-1}(0.75)^{1}(0.25)^{1}$$
$$+ \binom{3-1}{1-1}(0.75)^{1}(0.25)^{2} + \binom{4-1}{1-1}(0.75)^{1}(0.25)^{3}$$
$$+ \binom{5-1}{1-1}(0.75)^{1}(0.25)^{4}$$

$$= 0.75 + 0.1875 + 0.0469 + 0.0117 + 0.0029$$

$$P(X \leq 5) = \boxed{0.9990}$$

Remaining questions on next register

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Question (43)

A person throws a pair of fair dice. what is the probability that he will get a total of seven for the 2nd time on the eighth throw?

Solution:-

The Probability that he gets a total of 7 is $\frac{6}{36} \Rightarrow p = \frac{1}{6}$

Since it is a negative binomial di with $k = 2$, $x = 8$. so

$$P(X = x) = b^*(x; k, p) = \binom{x-1}{k-1} p^k q^{x-k}$$

$$= b^*\left(8; 2, \frac{1}{6}\right) = \binom{8-1}{2-1}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{8-2}$$

$$= \binom{7}{1}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^6$$

$$b^*\left(8; 2, \frac{1}{6}\right) = \boxed{0.0651}$$

Question (44)

The probability that a swimmer will succeed in swimming across a lake is 0.4. what is the probability that the tenth swimmer is fourth one to cross the lake?

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Solution:- 3/2

Since it is a negative binomial dist with $x = 10$, $k = 4$, $p = 0.4$, $q = 0.6$

$$b^*(x; k, p) = \binom{x-1}{k-1} p^k q^{x-k}$$

$$b^*(10, 4, 0.4) = \binom{10-1}{4-1} (0.4)^4 (0.6)^{10-4}$$

$$= \binom{9}{3} (0.4)^4 (0.6)^6$$

$$= \boxed{0.10033}$$

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Question (45)

If $E(X) = 10$, $\sigma = 3$ Can X have a negative binomial distribution?

Solution:-

Given that $E(X) = \text{Mean} = 10$

and $\sigma = 3 \Rightarrow Var(X) = \sigma^2 = 9$

Since Mean > Variance. So X Can not have a negative binomial distribution.

Question (46)

Find the probability that a person flipping a coin gets the third head on the seventh flip.

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Solution:-

Since it is negative binomial distribution with $x=7$, $k=3$, $p=\frac{1}{2}$

$$b^*(x;k,p) = \binom{x-1}{k-1} p^k q^{x-k}$$

$$b^*(7;3,\tfrac{1}{2}) = \binom{7-1}{3-1}\left(\tfrac{1}{2}\right)^3\left(\tfrac{1}{2}\right)^{7-3}$$

$$= \binom{6}{2}\left(\tfrac{1}{2}\right)^3\left(\tfrac{1}{2}\right)^4$$

$$= 0.1172$$

$$b^*(7;3,\tfrac{1}{2}) = \boxed{0.1172}$$

Question (47)

The probability that a person will install a black telephone in a residence is estimated to be 0.3 Find the probability that the tent phone installed in a new subdivisio is the 5th black phone.

Solution:-

Since it is negative binomial dist with $x=10$, $k=5$, $p=0.3$, $q=0.7$

$$b^*(x;k,p) = \binom{x-1}{k-1} p^k q^{x-k}$$

$$b^*(x;k,p) = \binom{10-1}{5-1}(0.3)^5(0.7)^{10-5}$$

$$b^*(10;5,0.3) = \boxed{0.0515}$$

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Question (48) (3.59)

The Probability that a person living in a certain city owns a dog is estimated to be 0.3

Find the probability that the tenth person randomly interviewed in this city is the fifth one to own a dog.

Solution:-

Since it is negative binomial dist with $x = 10$, $k = 5$, $p = 0.3$, $q = 0.7$

So

$$b^*(x; k, p) = \binom{x-1}{k-1} p^k q^{x-k}$$

$$b^*(10; 5, 0.3) = \binom{10-1}{5-1} (0.3)^5 (0.7)^{10-5}$$

$$= \binom{9}{4} (0.3)^5 (0.7)^5$$

$$b^*(10; 5, 0.3) = \boxed{0.05147}$$

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A/2010

Gamma Function.

The gamma function for any number $m>0$ denoted by Γm, is defined by

$$\Gamma(m) = \int_0^\infty x^{m-1} e^{-x}\, dx$$

Also $\Gamma(m) = (m-1)!$ or $\Gamma(m) = (m-1)\Gamma(m-1)$

Gamma Distribution.

A Continuous random variable X is said to have a gamma distribution with parameter $m>0$ if its p.d.f is

$$f(x) = \begin{cases} \dfrac{e^{-x} x^{m-1}}{\Gamma(m)} & \text{for } 0 \le x < \infty \\ 0 & ; \text{eleswhere} \end{cases}$$

Note.

$$\int_0^\infty f(x)\, dx = \int_0^\infty \frac{e^{-x} x^{m-1}}{\Gamma(m)}\, dx$$

$$= \frac{1}{\Gamma(m)} = \int_0^\infty e^{-x} x^{m-1}\, dx$$

$$= \frac{1}{\cancel{\Gamma(m)}} \cdot \cancel{\Gamma(m)} = 1$$

$\Rightarrow$ it is proper p.d.f.

Mean. A/2017

S/2012 $$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{-\infty}^{0} x f(x)\, dx + \int_0^\infty x f(x)\, dx$$

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$$= 0 + \int_0^\infty x \cdot \frac{e^{-x} x^{m-1}}{\Gamma(m)} dx$$

$$= \frac{1}{\Gamma(m)} \int_0^\infty e^{-x} x^{m+1-1} dx$$

$$= \frac{1}{\Gamma(m)} \Gamma(m+1) = \frac{m!}{(m-1)!}$$

$$= \frac{m\cancel{(m-1)!}}{\cancel{(m-1)!}} = m$$

$$\Rightarrow \boxed{E(X) = m}$$

Variance. S/2012 A/2017

$$V(X) = E(X^2) - (E(X))^2 \quad \text{——— (1)}$$

Note that

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$$

$$= \int_{-\infty}^{0} x^2 f(x) dx + \int_0^\infty x^2 f(x) dx$$

$$= 0 + \int_0^\infty \frac{x^2 e^{-x} x^{m-1}}{\Gamma(m)} dx$$

$$= \frac{1}{\Gamma(m)} \int_0^\infty e^{-x} x^{m+2-1} dx$$

$$= \frac{1}{\Gamma(m)} \Gamma(m+2)$$

$$= \frac{(m+1)!}{(m-1)!}$$

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$$= \frac{(m+1)(m)\cancel{(m-1)!}}{\cancel{(m-1)!}} = m(m+1)$$

Thus,

$$① \Rightarrow V(X) = m(m+1) - m^2$$

$$= \cancel{m^2} + m - \cancel{m^2}$$

$$\Rightarrow \boxed{V(X) = m}$$

Moments About origin.

$$\mu_r' = E(X^r)$$

$$= \int_{-\infty}^{\infty} x^r f(x)\,dx$$

$$= \int_{-\infty}^{0} x^r f(x)\,dx + \int_{0}^{\infty} x^r f(x)\,dx$$

$$= 0 + \int_{0}^{\infty} x^r \frac{e^{-x} x^{m-1}}{\Gamma(m)}\,dx$$

$$= \frac{1}{\Gamma(m)} \int_{0}^{\infty} x^{m+r-1} e^{-x}\,dx$$

$$= \frac{1}{\Gamma(m)} \Gamma(m+r)$$

$$\Rightarrow \boxed{\mu_r' = \frac{(m+r-1)!}{(m-1)!}}$$

$$\mu_1' = \frac{m!}{(m-1)!} = \frac{m\cancel{(m-1)!}}{\cancel{(m-1)!}} = m$$

$$\Rightarrow \boxed{\mu_1' = m}$$

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$$\mu_2' = \frac{(m+1)!}{(m-1)!} = \frac{m(m+1)\cancel{(m-1)!}}{\cancel{(m-1)!}}$$

$$\Rightarrow \boxed{\mu_2' = m(m+1)}$$

$$\mu_3' = \frac{(m+2)!}{(m-1)!} = \frac{m(m+1)(m+2)\cancel{(m-1)!}}{\cancel{(m-1)!}}$$

$$\Rightarrow \boxed{\mu_3' = m(m+1)(m+2)}$$

$$\mu_4' = \frac{(m+3)!}{(m-1)!} = \frac{(m+3)(m+2)(m+1)m\cancel{(m-1)!}}{\cancel{(m-1)!}}$$

$$\Rightarrow \boxed{\mu_4' = m(m+1)(m+2)(m+3)}$$

Moments About Mean.

$$\mu_r = E(X-\mu)^r$$
$$= E(X-m)^r$$

$$\mu_1 = E(X) - m = 0$$

$$\Rightarrow \mu_2 = \mu_2' - (\mu_1')^2 = m(m+1) - m^2$$

$$\Rightarrow \boxed{\mu_2 = m}$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$
$$= m(m+1)(m+2) - 3m(m+1)m + 2m^3$$
$$= (m^2+m)(m+2) - 3m^2(m+1) + 2m^3$$
$$= \cancel{m^3} + \cancel{2m^2} + \cancel{m^2} + 2m - \cancel{3m^3} - \cancel{3m^2} + \cancel{2m^3}$$

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$$\boxed{\mu_3 = 2m}$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$

$$= (m^2+m)(m^2+5m+6) - 4(m^3+2m^2+m^2+2m)m$$
$$+ 6(m^2+m)(m^2) - 3m^4$$

$$= \cancel{m^4} + \cancel{5m^3} + \cancel{6m^2} + \cancel{m^3} + 5m^2 + 6m - \cancel{4m^4} - \cancel{8m^3}$$
$$- \cancel{4m^3} - 8m^2 + \cancel{6m^4} + \cancel{6m^3} - \cancel{3m^4}$$

$$= 3m^2 + 6m$$

$$\Rightarrow \boxed{\mu_4 = 3m(m+2)}$$

A/2015 S/2012 A/2007

M.G.F. About origin.

$$M_0(t) = E(e^{tx})$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_{-\infty}^{0} e^{tx} f(x)\,dx + \int_{0}^{\infty} e^{tx} f(x)\,dx$$

$$= 0 + \int_{0}^{\infty} e^{tx} \frac{e^{-x} x^{m-1}}{\Gamma(m)}\,dx$$

$$= \frac{1}{\Gamma(m)} \int_{0}^{\infty} e^{-(1-t)x} x^{m-1}\,dx$$

Put $(1-t)x = u \Rightarrow (1-t)dx = du$

$$\Rightarrow dx = \frac{du}{1-t}$$

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when $x \to 0$, $u \to 0$

$x \to \infty$, $u \to \infty$

Then $M_0(t) = \frac{1}{\Gamma(m)} \int_0^\infty e^{-u} \left(\frac{u}{1-t}\right)^{m-1} \cdot \frac{du}{(1-t)}$

$$= \frac{1}{\Gamma(m)} \int_0^\infty e^{-u} u^{m-1} \cdot \frac{du}{(1-t)^{m-1+1}}$$

$$= \frac{1}{\Gamma(m)} \int_0^\infty \frac{e^{-u} u^{m-1}}{(1-t)^m} du$$

$$= \frac{1}{\Gamma(m)(1-t)^m} \int_0^\infty e^{-u} u^{m-1} du$$

$$= \frac{1}{\cancel{\Gamma(m)}(1-t)^m} \cancel{\Gamma(m)}$$

$$\Rightarrow M_0(t) = \frac{1}{(1-t)^m} = (1-t)^{-m}$$

$$\Rightarrow \boxed{M_0(t) = (1-t)^{-m}} \quad ; |t| < 1$$

$$\mu_r' = \frac{d^r}{dt^r} (1-t)^{-m} \Big|_{t=0}$$

$$\mu_1' = \frac{d}{dt} (1-t)^{-m} \Big|_{t=0}$$

$$= -m(1-t)^{-m-1} (-1) \Big|_{t=0}$$

$$= m(1-t)^{-m-1} \Big|_{t=0}$$

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$\Rightarrow \mu_1' = m$

$$\mu_2' = \frac{d}{dt} m(1-t)^{-m-1}\Big|_{t=0}$$

$$= m(-m-1)(1-t)^{-m-2}(-1)\Big|_{t=0}$$

$$= m(m+1)(1-t)^{-m-2}\Big|_{t=0}$$

$$\Rightarrow \mu_2' = m(m+1)$$

$$\mu_3' = \frac{d}{dt} m(m+1)(1-t)^{-m-2}\Big|_{t=0}$$

$$= m(m+1)(-m-2)(1-t)^{-m-3}(-1)\Big|_{t=0}$$

$$= m(m+1)(m+2)$$

$$\mu_4' = \frac{d}{dt} m(m+1)(m+2)(1-t)^{-m-3}\Big|_{t=0}$$

$$= m(m+1)(m+2)(-m-3)(1-t)^{-m-4}(-1)\Big|_{t=0}$$

$$= m(m+1)(m+2)(m+3)$$

M.G.F. About Mean. A/2014

$$M_{\mu}(t) = E(e^{t(x-m)})$$

$$= \int_{-\infty}^{\infty} e^{t(x-m)} f(x)\,dx.$$

$$= \int_{-\infty}^{0} e^{t(x-m)} f(x)dx + \int_{0}^{\infty} e^{tx} e^{-mt} f(x)dx$$

$$= e^{-mt} \int_{0}^{\infty} e^{tx} \frac{e^{-x} x^{m-1}}{\Gamma(m)} dx$$

$$= e^{-mt}(1-t)^{-m}$$

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$$= e^{-mt}(1-t)^{-m}$$

$$= [e^{t}(1-t)]^{-m}$$

$$\Rightarrow \boxed{M_x(t) = (e^{t} - te^{t})^{-m}}$$

$$\mu_1 = \frac{d^1}{dt^1}(e^{t} - te^{t})^{-m}\Big|_{t=0}$$

$$\mu_1 = \frac{d}{dt}(e^{t} - te^{t})^{-m}\Big|_{t=0}$$

$$= -m(e^{t} - te^{t})^{-m-1}(\cancel{e^{t}} - \cancel{e^{t}} - te^{t})\Big|_{t=0}$$

$$= mte^{t}(e^{t} - te^{t})^{-m-1}\Big|_{t=0}$$

$$\Rightarrow \mu_1 = 0$$

$$\mu_2 = \frac{d}{dt} mte^{t}(e^{t} - te^{t})^{-m-1}\Big|_{t=0}$$

$$= m\{e^{t}(e^{t} - te^{t})^{-m-1} + te^{t}(e^{t} - te^{t})^{-m-1}$$

$$(-m-1)te^{t}(e^{t} - te^{t})^{-m-2}(\cancel{e^{t}} - \cancel{e^{t}} - te^{t})\}_{t=0}$$

$$= m\{e^{t}(e^{t} - te^{t})^{-m-1} + te^{t}(e^{t} - te^{t})^{-m-1}$$

$$+ (m+1)t^2e^{2t}(e^{t} - te^{t})^{-m-2}\}_{t=0}$$

$$= m\{(e^{t} - te^{t})^{-m-1}(e^{t} + te^{t})$$

$$+ (m+1)t^2e^{2t}(e^{t} - te^{t})^{-m-2}\}_{t=0}$$

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$$= m\{1+0\} = m$$

$$\mu_3 = \frac{d}{dt}\left\{m(e^t - te^t)^{-m-1}(e^t + te^t) + m(m+1)t^2 e^{2t}(e^t - te^t)^{-m-2}\right\}_{t=0}$$

$$= \Big[m(e^t - te^t)^{-m-1}(e^t + e^t + te^t) + m(-m-1)(e^t - te^t)^{-m-2}(-te^t)(e^t + te^t) + m(m+1)\{2te^{2t}(e^t - te^t)^{-m-2} + 2t^2e^{2t}(e^t - te^t)^{-m-2} + t^2e^{2t}(e^t - te^t)^{-m-3}(-m-2)(-te^t)\Big]_{t=0}$$

$$= \Big[m(e^t - te^t)^{-m-1}(2e^t + te^t) + m(m+1)te^t(e^t - te^t)^{-m-2}(e^t + te^t) - 2m(m+1)te^{2t}(e^t - te^t)^{-m-2} + 2m(m+1)t^2e^{2t}(e^t - te^t)^{-m-2} + m(m+1)(m+2)t^3e^{3t}(e^t - te^t)^{-m-3}\Big]_{t=0}$$

$$= \Big[m(e^t - te^t)^{-m-1}(2e^t + te^t) + m(m+1)te^{2t}(e^t - te^t)^{-m-2} + m(m+1)t^2e^{2t}(e^t - te^t)^{-m-2} + 2m(m+1)te^{2t}(e^t - te^t)^{-m-2} + 2m(m+1)t^2e^{2t}(e^t - te^t)^{-m-2} + m(m+1)(m+2)t^3e^{3t}(e^t - te^t)^{-m-3}\Big]_{t=0}$$

$$= \Big[m(e^t - te^t)^{-m-1}(2e^t + te^t) + m(m+1)\{te^{2t} + 3t^2e^{2t}\}(e^t - te^t)^{-m-2} + m(m+1)(m+2)t^3e^{3t}(e^t - te^t)^{-m-3}\Big]_{t=0}$$

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$$= m(1)(2) + m(m+1)(0)(1) + m(m+1)(m+2)(0)$$

$$= 2m + 0 + 0$$

$$\Rightarrow \mu_3 = 2m$$

$$\mu_4 = \frac{d}{dt}\Big[m(e^t - te^t)^{-m-1}(2+t)e^t + 3m(m+1)(te^{2t} + t^2e^{2t})(e^t - te^t)^{-m-2} + m(m+1)(m+2)\,t^3e^{3t}(e^t - te^t)^{-m-3}\Big]_{t=0}$$

$$= \Big[m(e^t - te^t)^{-m-1}(2e^t + e^t + te^t) + m(-m-1)(e^t - te^t)^{-m-1}(-te^t)(2+t)e^t + 3m(m+1)\{(e^{2t} + 2te^{2t} + 2te^{2t} + 2t^2e^{2t})(e^t - te^t)^{-m-2} + (-m-2)(e^t - te^t)^{-m-3}(-te^t)(te^{2t} + t^2e^{2t})\} + m(m+1)(m+2)\{3t^2e^{3t}(e^t - te^t)^{-m-3} + 3t^3e^{3t}(e^t - te^t)^{-m-3} + (-m-3)t^3e^{3t}(e^t - te^t)^{-m-4}(-te^t)\}\Big]_{t=0}$$

$$= m(1)(3) + m(m+1)(0) + 3m(m+1)(1) + m(m+1)(m+2)(0)$$

$$= 3m + 3m(m+1)$$

$$= 3m + 3m^2 + 3m$$

$$= 3m^2 + 6m$$

$$\Rightarrow \mu_4 = 3m(m+2).$$

A/2014
Find. β_1 and β_2

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Cumulant Generating Function.

The cumulant generating function is given by

$$K(t) = \ln M_0(t)$$
$$= \ln (1-t)^{-m} \quad ; \; |t| > 1$$
$$= -m \ln (1-t)$$
$$= -m\left\{-t - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{4} - \cdots\right\}$$

$$\Rightarrow K(t) = mt + \frac{mt^2}{2} + \frac{mt^3}{3} + \frac{mt^4}{4} + \cdots$$

$$= m\frac{t}{1!} + m\frac{t^2}{2!} + 2m\frac{t^3}{3!} + 6m\frac{t^4}{4!} + \cdots \rightarrow (1)$$

The general expression for c.g.f is

$$K(t) = K_1\frac{t}{1!} + K_2\frac{t^2}{2!} + K_2\frac{t^3}{3!} + K_3\frac{t^4}{4!} + \cdots \rightarrow (2)$$

Comparing like powers of t in (1) and (2), we get

$K_1 = m \quad ; \quad K_2 = m$

$K_3 = 2m \quad ; \quad K_4 = 6m$

Harmonic mean.

$$H.M = \frac{1}{E(1/x)} \quad \text{———} \quad (1)$$

where $E\left(\frac{1}{x}\right) = \int_{-\infty}^{\infty} \frac{1}{x} f(x)\,dx$

$$= \int_{-\infty}^{0} \frac{1}{x} f(x)\,dx + \int_{0}^{\infty} \frac{1}{x} f(x)\,dx$$

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$$= 0 + \int_0^\infty \frac{1}{x} \frac{e^{-x} x^{m-1}}{\Gamma(m)} dx$$

$$= \frac{1}{\Gamma(m)} \int_0^\infty e^{-x} x^{m-2} dx$$

$$= \frac{1}{\Gamma(m)} \int_0^\infty e^{-x} x^{m-1-1} dx$$

$$= \frac{1}{\Gamma(m)} \Gamma(m-1)$$

$$= \frac{(m-2)!}{(m-1)!} = \frac{\cancel{(m-2)!}}{(m-1)\cancel{(m-2)!}}$$

$$\Rightarrow E\left(\frac{1}{X}\right) = \frac{1}{m-1}$$

Now eq ① becomes

$$\boxed{H.M = m-1}$$

Mode. A/2010, A/2013,

The mode of a continuous r.v X is such a stationary value of $f(x)$ for which the p.d.f. is maximum.

Here $f(x) = \frac{e^{-x} x^{m-1}}{\Gamma(m)}$; $0 \le x < \infty$

Diff. w.r.t. x, we get

$$f'(x) = \frac{1}{\Gamma(m)} \left\{ e^{-x} (m-1) x^{m-2} + e^{-x} (-1) x^{m-1} \right\}$$

$$= \frac{1}{\Gamma(m)} \left\{ (m-1) e^{-x} x^{m-2} - e^{-x} x^{m-1} \right\} \quad ①$$

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Put $f'(x) = 0$

then $\frac{1}{\Gamma(m)}\{(m-1)e^{-x}x^{m-2} - e^{-x}x^{m-1}\} = 0$

$$\Rightarrow \frac{e^{-x}x^{m-1}}{\Gamma(m)}\{(m-1)x^{-1} - 1\} = 0$$

But $\frac{e^{-x}x^{m-1}}{\Gamma(m)} \neq 0$

so $\frac{m-1}{x} - 1 = 0$

$$\Rightarrow \frac{m-1}{x} = 1 \Rightarrow \boxed{m-1 = x}$$

Diff. ① w.r.t. x, we get

$$f''(x) = \frac{1}{\Gamma(m)}\{(m-1)((m-2)e^{-x}x^{m-3} - e^{-x}x^{m-2}) - ((m-1)e^{-x}x^{m-2} - e^{-x}x^{m-1})\}$$

$$= \frac{1}{\Gamma(m)}\{(m-1)(m-2)e^{-x}x^{m-3} - (m-1)e^{-x}x^{m-2} - (m-1)e^{-x}x^{m-2} + e^{-x}x^{m-1}\}$$

$$= \frac{1}{\Gamma(m)}\{(m-1)(m-2)e^{-x}x^{m-3} - 2(m-1)e^{-x}x^{m-2} + e^{-x}x^{m-1}\}$$

$$= \frac{e^{-x}x^{m-1}}{\Gamma(m)} * \{(m-1)(m-2)x^{-2} - 2(m-1)x^{-1} + 1\}$$

$$= \frac{e^{-x}x^{m-1}}{\Gamma(m)}\left\{\frac{(m-1)(m-2)}{x^2} - \frac{2(m-1)}{x} + 1\right\}$$

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Putting the value of x, we get

$$f''(m-1) = \frac{e^{-(m-1)}(m-1)^{m-1}}{\Gamma(m)}\left\{\frac{\cancel{(m-1)}(m-2)}{(m-1)^{\cancel{2}}} - 2\frac{\cancel{(m-1)}}{\cancel{m-1}} + 1\right\}$$

$$= \frac{e^{1-m}(m-1)^{m-1}}{\Gamma(m)}\left\{\frac{m-2}{m-1} - 1\right\}$$

$$= \frac{e^{1-m}(m-1)^{m-1}}{\Gamma(m)}\left\{\frac{\cancel{m}-2-\cancel{m}+1}{m-1}\right\}$$

$$= -\frac{e^{1-m}(m-1)^{m-1}}{(m-1)\Gamma(m)}$$

$$= -\frac{e^{1-m}(m-1)^{m-2}}{\Gamma(m)} < 0$$

Hence $x = m-1$ is mode.

Median.

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Beta Function.

The beta function for any two +ve numbers m, n, denoted by $\beta(m,n)$, is defined by

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad \text{for } m,n>0$$

Beta Distribution of first kind.

A Continuous random variable X is said to have a beta distribution of the 1st kind with two parameters m and n, if its p.d.f is

$$f(x) = \begin{cases} \dfrac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1} & ; 0 \le x \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

It is denoted by $\beta_1(m,n)$.

Note.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^1 f(x)\,dx + \int_1^{\infty} f(x)\,dx$$

$$= \int_0^1 \frac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1}\,dx$$

$$= \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{\beta(m,n)}\,dx$$

$$= \frac{1}{\beta(m,n)} \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= \frac{1}{\cancel{\beta(m,n)}} \cancel{\beta(m,n)} = 1.$$

$\Rightarrow \beta_1$ is proper pdf

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Mean. A/2018

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_{-\infty}^{0} x f(x)\,dx + \int_{0}^{1} x f(x)\,dx + \int_{1}^{\infty} x f(x)\,dx$$

$$= \int_{0}^{1} x \cdot \frac{1}{\beta(m,n)} x^{m-1} (1-x)^{n-1}\,dx$$

$$= \frac{1}{\beta(m,n)} \int_{0}^{1} x^{m+1-1} (1-x)^{n-1}\,dx$$

$$= \frac{1}{\beta(m,n)} \cdot \beta(m+1, n)$$

Since, we know that

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Therefore.

$$E(X) = \frac{\Gamma(m+n)}{\Gamma(m)\cancel{\Gamma(n)}} \cdot \frac{\Gamma(m+1)\cancel{\Gamma(n)}}{\Gamma(m+1+n)}$$

$$= \frac{\cancel{\Gamma(m+n)}}{\cancel{\Gamma(m)}} \cdot \frac{m\cancel{\Gamma(m)}}{(m+n)\cancel{\Gamma(m+n)}}$$

$$\Rightarrow \boxed{E(X) = \frac{m}{m+n}}$$

A/2018

Variance.

$$V(X) = E(X^2) - (E(X))^2 \quad \text{——— (1)}$$

Note that

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx.$$

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$$= \int_0^1 x^2 \frac{1}{\beta(m,n)} x^{m-1} (1-x)^{n-1} dx$$

$$= \frac{1}{\beta(m,n)} \int_0^1 x^{m+2-1} (1-x)^{n-1} dx$$

$$= \frac{1}{\beta(m,n)} \cdot \beta(m+2, n)$$

$$= \frac{\Gamma(m+n)}{\Gamma(m)\cancel{\Gamma(n)}} \cdot \frac{\Gamma(m+2)\cancel{\Gamma(n)}}{\Gamma(m+2+n)}$$

$$= \frac{\Gamma(m+n)}{\Gamma(m)} \times \frac{(m+1)\Gamma(m+1)}{(m+n+1)\Gamma(m+n+1)}$$

$$= \frac{\cancel{\Gamma(m+n)}}{\cancel{\Gamma(m)}} \times \frac{(m+1)(m)\cancel{\Gamma(m)}}{(m+n+1)(m+n)\cancel{\Gamma(m+n)}}$$

$$= \frac{m(m+1)}{(m+n)(m+n+1)}$$

① $\Rightarrow$ $$V(X) = \frac{m(m+1)}{(m+n)(m+n+1)} - \frac{m^2}{(m+n)^2}$$

$$= \frac{(m+n)(m^2+m) - m^2(m+n+1)}{(m+n)^2(m+n+1)}$$

$$= \cancel{\frac{m^2 + m^2 - m^2 - mn - m}{(m+n)(m+n+1)}}$$

$$= \frac{\cancel{m^3} + \cancel{m^2} + \cancel{m^2n} + mn - \cancel{m^3} - \cancel{m^2n} - \cancel{m^2}}{(m+n)^2(m+n+1)}$$

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$$\boxed{V(x) = \frac{mn}{(m+n)^2(m+n+1)}}$$

Moments About origin.

$$\mu_r' = E(x^r)$$

$$= \int_{-\infty}^{\infty} x^r \, f(x)\, dx.$$

$$= \int_0^1 x^r \frac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1} dx$$

$$= \frac{1}{\beta(m,n)} \int_0^1 x^{m+r-1}(1-x)^{n-1} dx$$

$$= \frac{1}{\beta(m,n)} \cdot \beta(m+r, n)$$

$$= \frac{\Gamma(m+n)}{\Gamma(m)\cancel{\Gamma(n)}} \times \frac{\Gamma(m+r)\cancel{\Gamma(n)}}{\Gamma(m+r+n)}$$

$$= \frac{\Gamma(m+n)}{\Gamma(m)} \times \frac{\Gamma(m+r)}{\Gamma(m+r+n)}$$

$$= \frac{\cancel{\Gamma(m+n)}}{\cancel{\Gamma(m)}} \times \frac{(m+r-1)(m+r-2)\cdots m\,\cancel{\Gamma(m)}}{(m+n+r-1)(m+n+r-2)\cdots(m+n)\cancel{\Gamma(m+n)}}$$

$$= \frac{(m+r-1)(m+r-2)\cdots m}{(m+n+r-1)(m+n+r-2)\cdots(m+n)}$$

$$\mu_1' = \frac{\cancel{m(m+1)}}{\cancel{(m+n)(m+n+1)}} \qquad \mu_1' = \frac{m}{m+n}$$

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$$\mu_2' = \frac{m(m+1)}{(m+n+1)(m+n)}$$

$$\mu_3' = \frac{m(m+1)(m+2)}{(m+n)(m+n+1)(m+n+2)}$$

and

$$\mu_4' = \frac{m(m+1)(m+2)(m+3)}{(m+n)(m+n+1)(m+n+2)(m+n+3)}$$

Harmonic Mean.

$$H \cdot M = \frac{1}{E(\frac{1}{x})} \quad \text{——— (1)}$$

where

$$E\left(\frac{1}{x}\right) = \int_{-\infty}^{\infty} \frac{1}{x} f(x)\, dx.$$

$$= \int_0^1 \frac{1}{x} \frac{1}{\beta(m,n)} x^{m-1} (1-x)^{n-1} dx$$

$$= \frac{1}{\beta(m,n)} \int_0^1 x^{m-1-1} (1-x)^{n-1} dx.$$

$$= \frac{1}{\beta(m,n)} \cdot \beta(m-1, n)$$

$$= \frac{\Gamma(m+n)}{\Gamma(m)\cancel{\Gamma(n)}} \cdot \frac{\Gamma(m-1)\cancel{\Gamma(n)}}{\Gamma(m-1+n)}$$

$$= \frac{\Gamma(m+n)}{\Gamma(m)} \cdot \frac{\Gamma(m-1)}{\Gamma(m+n-1)}$$

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$$= \frac{(m+n-1)\,\cancel{\Gamma(m+n-1)}}{(m-1)\,\cancel{\Gamma(m-1)}} \cdot \frac{\cancel{\Gamma(m-1)}}{\cancel{\Gamma(m+n-1)}}$$

$$= \frac{m+n-1}{m-1}$$

Thus equation ① becomes

$$\boxed{H.M = \frac{m-1}{m+n-1}}$$

Mode.

A/2013

Here $f(x) = \frac{1}{\beta(m,n)} x^{m-1} (1-x)^{n-1}$; $0 \le x < 1$

Diff. w.r.t. x, we get

$$f'(x) = \frac{1}{\beta(m,n)} \left\{ x^{m-1}(n-1)(1-x)^{n-2}(-1) + (m-1)x^{m-2}(1-x)^{n-1} \right\}$$

$$= \frac{1}{\beta(m,n)} \left\{ -(n-1)x^{m-1}(1-x)^{n-2} + (m-1)x^{m-2}(1-x)^{n-1} \right\}$$

$$= \frac{x^{m-1}(1-x)^{n-1}}{\beta(m,n)} \left\{ (1-n)(1-x)^{-1} + (m-1)x^{-1} \right\}$$

$$= \frac{x^{m-1}(1-x)^{n-1}}{\beta(m,n)} \left[\frac{1-n}{1-x} + \frac{m-1}{x} \right] \quad \text{①}$$

Put $f'(x) = 0$

Then $\frac{x^{m-1}(1-x)^{n-1}}{\beta(m,n)} \left[\frac{1-n}{1-x} + \frac{m-1}{x} \right] = 0$

But $\frac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1} \neq 0$

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So $\frac{1-n}{1-x} + \frac{m-1}{x} = 0$

$$\Rightarrow \frac{1-n}{1-x} = \frac{1-m}{x}$$

$$\Rightarrow (1-n)x = (1-m)(1-x)$$

$$\Rightarrow x(1-n) = 1 - x - m + mx$$

$$\Rightarrow x - nx = 1 - x - m + mx$$

$$\Rightarrow x - nx - 1 + x + m - mx = 0$$

$$\Rightarrow 2x - nx + m - mx - 1 = 0$$

$$\Rightarrow 2x - nx - mx = 1 - m$$

$$\Rightarrow -x(m+n-2) = 1 - m$$

$$\Rightarrow \boxed{x = \frac{m-1}{m+n-2}}$$

Diff. $f'(x)$, w.r.t. x, we get

$$f''(x) = \frac{1}{\beta(m,n)} \frac{d}{dx}\left[(1-n)x^{m-1}(1-x)^{n-2} + (m-1)x^{m-2}(1-x)^{n-1}\right]$$

$$= \frac{1}{\beta(m,n)}\Big[(1-n)\{x^{m-1}(n-2)(1-x)^{n-3}(-1) + (m-1)x^{m-2}(1-x)^{n-2}\}$$
$$+ (m-1)\{x^{m-2}(n-1)(1-x)^{n-2}(-1) + (m-2)x^{m-3}(1-x)^{n-1}\}\Big]$$

$$= \frac{1}{\beta(m,n)}\Big[(1-n)(2-n)x^{m-1}(1-x)^{n-3} + (1-n)(m-1)x^{m-2}(1-x)^{n-2}$$
$$+ (m-1)(1-n)x^{m-2}(1-x)^{n-2} + (m-1)(m-2)x^{m-3}(1-x)^{n-1}\Big]$$

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$$= \frac{1}{\beta(m,n)}\Big[(1-n)(2-n)x^{m-1}(1-x)^{n-3} + 2(1-n)(m-1)x^{m-2}(1-x)^{n-2} + (m-1)(m-2)(x)^{m-3}(1-x)^{n-1}\Big]$$

Putting the value of x in $f''(x)$, we have

$$f''(x)\Big|_{x=\frac{m-1}{m+n-2}} = \frac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1}\left[\frac{(1-n)(2-n)}{(1-x)^2} + \frac{2(1-n)(m-1)}{x(1-x)} + \frac{(m-1)(m-2)}{x^2}\right]_{x=\frac{m-1}{m+n-2}} \quad (*)$$

Now,

$$1-x = 1 - \frac{m-1}{m+n-2}$$

$$= \frac{\cancel{m}+n-2-\cancel{m}+1}{m+n-2} = \frac{n-1}{m+n-2}$$

and

$$x(1-x) = \frac{m-1}{m+n-2}\left(\frac{n-1}{m+n-2}\right)$$

$$= \frac{(m-1)(n-1)}{(m+n-2)^2}$$

Consider

$$\frac{(1-n)(2-n)}{(1-x)^2} + \frac{2(1-n)(m-1)}{x(1-x)} + \frac{(m-1)(m-2)}{x^2}$$

$$= \frac{(1-n)(2-n)}{(n-1)^2} \times (m+n-2)^2 + \frac{2(1-n)\cancel{(m-1)}}{\cancel{(m-1)}(n-1)}(m+n-2)^2$$

$$+ \frac{\cancel{(m-1)}(m-2)}{(m-1)^{\cancel{2}}} \times (m+n-2)^2$$

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$$= -\frac{(2-n)}{n-1}(m+n-2)^2 - 2(m+n-2)^2 + \frac{m-2}{m-1}(m+n-2)^2$$

$$= (m+n-2)^2\left[\frac{n-2}{n-1} - 2 + \frac{m-2}{m-1}\right]$$

$$= (m+n-2)^2\left[\frac{(n-2)(m-1) - 2(n-1)(m-1) + (m-2)(n-1)}{(n-1)(m-1)}\right]$$

$$= \frac{(m+n-2)^2}{(m-1)(n-1)}\left[mn - n - 2m + 2 - 2(mn - n - m + 1) + mn - m - 2n + 2\right]$$

$$= \frac{(m+n-2)^2}{(m-1)(n-1)}\left[\cancel{2mn} - 3n - 3m + 4 \cancel{-2mn} + 2n + 2m - 2\right]$$

$$= \frac{(m+n-2)^2}{(m-1)(n-1)}(-n - m + 2)$$

$$= \frac{(m+n-2)^2}{(m-1)(n-1)}\left[-(m+n-2)\right]$$

$$= -\frac{(m+n-2)^3}{(m-1)(n-1)}$$

Now eq ⊛ becomes

$$f''(x)\Big|_{x=\frac{m-1}{m+n-2}} = \frac{1}{\beta(m,n)}\left(\frac{m-1}{m+n-2}\right)^{m-1}\left(\frac{n-1}{m+n-2}\right)^{n-1}\left[\frac{-(m+n-2)^3}{(m-1)(n-1)}\right]$$

$$= -\frac{1}{\beta(m,n)} \cdot \frac{(m-1)^{m-2}(n-1)^{n-2}}{(m+n-2)^{m+n-5}} < 0$$

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Hence $x = \dfrac{m-1}{m+n-2}$ is mode.

Beta Distribution of Second Kind.

A continuous r.v. X is said to have a beta distribution of the 2nd kind with parameters m and n, if its p.d.f is defined by

$$f(x) = \begin{cases} \dfrac{1}{\beta(m,n)} \dfrac{x^{m-1}}{(1+x)^{m+n}} & ; 0 \le x < \infty \\ 0 & \text{elsewhere} \end{cases}$$

Note.

We will prove that β_2 is proper p.d.f.

We know that

$$\frac{1}{\beta(m,n)} \int_0^1 x^{m-1} (1-x)^{n-1} dx = 1$$

Put $u = \dfrac{x}{1-x} \Rightarrow u - ux = x$

$$\Rightarrow x + ux = u$$

$$\Rightarrow x = \frac{u}{1+u}$$

$$\Rightarrow dx = \frac{(1+u)(1) - u(1)}{(1+u)^2} du = \frac{1+\cancel{u}-\cancel{u}}{(1+u)^2} du$$

$$\Rightarrow dx = \frac{du}{(1+u)^2}$$

when $x \to 0$, $u \to 0$

$x \to 1$ $\quad u \to \infty$

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Then $\frac{1}{\beta(m,n)} \int_0^\infty \left(\frac{u}{1+u}\right)^{m-1} \left(1-\frac{u}{1+u}\right)^{n-1} \frac{du}{(1+u)^2} = 1$

$$\Rightarrow \frac{1}{\beta(m,n)} \int_0^\infty \frac{u^{m-1}}{(1+u)^{m-1}} \times \frac{1}{(1+u)^{n-1}} \frac{du}{(1+u)^2} = 1$$

$$\Rightarrow \frac{1}{\beta(m,n)} \int_0^\infty \frac{u^{m-1}}{(1+u)^{m+n}} du = 1$$

$$\Rightarrow \int_0^\infty \frac{1}{\beta(m,n)} \frac{u^{m-1}}{(1+u)^{m+n}} du = 1$$

$$\Rightarrow \int_0^\infty \frac{1}{\beta(m,n)} \frac{x^{m-1}}{(1+x)^{m+n}} dx = 1$$ dummy index

$$\Rightarrow \int_0^\infty f(x)\, dx = 1$$

$\Rightarrow \beta_2$ is proper p.d.f.

Mean.

$$E(x) = \int_{-\infty}^{\infty} x f(x)\, dx.$$

$$= \int_{-\infty}^{0} x f(x)\, dx + \int_0^\infty x f(x)\, dx.$$

$$= \int_0^\infty x \frac{1}{\beta(m,n)} \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

$$= \frac{1}{\beta(m,n)} \int_0^\infty \frac{x^{m+1-1}}{(1+x)^{m+n}} dx$$

Let $(1+x) = \frac{1}{y} \Rightarrow x = \frac{1}{y} - 1 = \frac{1-y}{y}$

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$$\Rightarrow y = \frac{1}{1+x}$$

$$\Rightarrow dx = \frac{-1}{y^2}\,dy$$

when $x \to 0 \quad \Rightarrow y \to 1$

$x \to \infty \quad \Rightarrow y \to 0$

Then $$E(X) = \frac{1}{\beta(m,n)} \int_1^0 \left(\frac{1-y}{y}\right)^m y^{m+n} \left(-\frac{1}{y^2}\right) dy$$

$$= \frac{1}{\beta(m,n)} \int_0^1 \frac{(1-y)^m}{y^m} y^{m+n} y^{-2}\, dy$$

$$= \frac{1}{\beta(m,n)} \int_0^1 \frac{(1-y)^m}{\cancel{y^m}} \cancel{y^m} y^n y^{-2}\, dy$$

$$= \frac{1}{\beta(m,n)} \int_0^1 (1-y)^m y^{n-2}\, dy$$

$$= \frac{1}{\beta(m,n)} \int_0^1 (1-y)^{m+1-1} y^{n-1-1}\, dy$$

$$= \frac{1}{\beta(m,n)} \beta(n-1, m+1)$$

$$= \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} \times \frac{\Gamma(n-1)\,\Gamma(m+1)}{\Gamma(n-1+m+1)}$$

$$= \frac{\cancel{\Gamma(m+n)}}{\cancel{\Gamma(m)}\Gamma(n)} \times \frac{\Gamma(n-1)\, m\, \cancel{\Gamma(m)}}{\cancel{\Gamma(m+n)}}$$

$$= \frac{m\,\cancel{\Gamma(n-1)}}{(n-1)\,\cancel{\Gamma(n-1)}} = \frac{m}{n-1}$$

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$$\Rightarrow \boxed{E(X) = \frac{m}{n-1}}$$

Variance. A/2010

$$V(X) = E(X^2) - (E(X))^2 \quad \text{——— (1)}$$

where

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_0^{\infty} x^2 \frac{1}{\beta(m,n)} \cdot \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$= \frac{1}{\beta(m,n)} \int_0^{\infty} \frac{x^{m+1}}{(1+x)^{m+n}}\,dx$$

Let $y = \frac{1}{1+x} \Rightarrow 1+x = \frac{1}{y} \Rightarrow dx = -\frac{1}{y^2}\,dy$

$\Rightarrow x = \frac{1}{y} - 1 = \frac{1-y}{y}$

$x \to 0 \Rightarrow y \to 1$

$x \to \infty \Rightarrow y \to 0$

Then $E(X^2) = \frac{1}{\beta(m,n)} \int_1^0 \left(\frac{1-y}{y}\right)^{m+1} y^{m+n} \left(-\frac{1}{y^2}\right) dy$

$$= \frac{1}{\beta(m,n)} \int_0^1 \frac{(1-y)^{m+1}}{y^{m+1}}\, y^{m+n} y^{-2}\,dy$$

$$= \frac{1}{\beta(m,n)} \int_0^1 \frac{(1-y)^{m+1}}{\cancel{y^m}\, y} \cdot \cancel{y^m} y^n y^{-2}\,dy$$

$$= \frac{1}{\beta(m,n)} \int_0^1 y^{n-3} (1-y)^{m+1}\,dy$$

$$= \frac{1}{\beta(m,n)} \int_0^1 y^{n-2-1} (1-y)^{m+2-1}\,dy$$

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$$= \frac{1}{\beta(m,n)} \cdot \beta(n-2, m+2)$$

$$= \frac{\cancel{\Gamma(m+n)}}{\Gamma(m)\Gamma(n)} \times \frac{\Gamma(n-2)\Gamma(m+2)}{\cancel{\Gamma(m+n)}}$$

$$= \frac{\Gamma(n-2)(m+1)(m)\cancel{\Gamma(m)}}{\cancel{\Gamma(m)}\,\Gamma(n)}$$

$$= m(m+1)\frac{\cancel{\Gamma(n-2)}}{(n-1)(n-2)\cancel{\Gamma(n-2)}}$$

$$= \frac{m(m+1)}{(n-1)(n-2)}$$

Now equation ① becomes

$$V(X) = \frac{m(m+1)}{(n-1)(n-2)} - \frac{m^2}{(n-1)^2}$$

$$= \frac{m(m+1)(n-1) - m^2(n-2)}{(n-1)^2(n-2)}$$

$$= \frac{(m^2+m)(n-1) - m^2n + 2m^2}{(n-1)^2(n-2)}$$

$$= \frac{\cancel{m^2n} - m^2 + mn - m - \cancel{m^2n} + 2m^2}{(n-1)^2(n-2)}$$

$$= \frac{m^2 + mn - n}{(n-1)^2(n-2)}$$

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$$\Rightarrow \boxed{V(x) = \frac{m(m+n-1)}{(n-1)^2(n-2)}}$$

Moments About Origin.

$$\mu_r' = E(X^r)$$

$$= \int_{-\infty}^{\infty} x^r f(x)\,dx$$

$$= \int_{-\infty}^{0} x^r f(x)\,dx + \int_{0}^{\infty} x^r f(x)\,dx$$

$$= \int_{0}^{\infty} x^r \frac{1}{\beta(m,n)} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$= \frac{1}{\beta(m,n)} \int_{0}^{\infty} \frac{x^{m+r-1}}{(1+x)^{m+n}}\,dx$$

Let $\frac{1}{1+x} = y \Rightarrow \frac{1}{y} = 1+x$

$$\Rightarrow x = \frac{1}{y} - 1 = \frac{1-y}{y}$$

$$dx = -1/y^2\,dy$$

when $x \to 0$, $y \to 1$

$x \to \infty$, $y \to 0$

Then

$$\mu_r' = \frac{1}{\beta(m,n)} \int_{1}^{0} \left(\frac{1-y}{y}\right)^{m+r-1} y^{m+n} \left(-\frac{1}{y^2}\right) dy$$

$$= 1/\beta(m,n) \int_{0}^{1} \frac{(1-y)^{m+r-1}}{y^{m+r-1}}\, y^m y^n y^{-2}\,dy$$

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$$= \frac{1}{\beta(m,n)} \int_0^1 \frac{(1-y)^{m+s-1}}{\cancel{y^m} y^{s-1}} \cancel{y^m} \cdot y^{n-2} \, dy$$

$$= \frac{1}{\beta(m,n)} \int_0^1 y^{n-2} \, y^{1-s} \, (1-y)^{m+s-1} \, dy$$

$$= \frac{1}{\beta(m,n)} \int_0^1 y^{n-s-1} \, (1-y)^{m+s-1} \, dy$$

$$= \frac{1}{\beta(m,n)} \beta(n-s, m+s)$$

$$= \frac{\cancel{\Gamma(m+n)}}{\Gamma(m)\Gamma(n)} \times \frac{\Gamma(n-s)\,\Gamma(m+s)}{\cancel{\Gamma(m+n)}}$$

$$= \frac{\Gamma(n-s)\,\Gamma(m+s)}{\Gamma(m)\,\Gamma(n)}$$

$$= \frac{(m+s-1)(m+s-2)\cdots m\,\cancel{\Gamma(m)}\,\cancel{\Gamma(n-s)}}{(n-1)(n-2)\cdots(n-s)\,\cancel{\Gamma(n-s)}\,\cancel{\Gamma(m)}}$$

$$= \frac{m(m+1)(m+2)\cdots(m+s-1)}{(n-1)(n-2)\cdots(n-s)}$$

$$\mu_1' = \frac{m}{n-1}$$

$$\mu_2' = \frac{m(m+1)}{(n-1)(n-2)}$$

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$$\mu_3' = \frac{m(m+1)(m+2)}{(n-1)(n-2)(n-3)}$$

$$\mu_4' = \frac{m(m+1)(m+2)(m+3)}{(n-1)(n-2)(n-3)(n-4)}$$

Harmonic Mean. N/2010

$$H.M = \frac{1}{E(1/x)} \quad \text{———} \quad (1)$$

where $E\left(\frac{1}{x}\right) = \int_{-\infty}^{\infty} \frac{1}{x} f(x)\,dx.$

$$= \int_0^{\infty} \frac{1}{x} \cdot \frac{1}{\beta(m,n)} \cdot \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^{\infty} \frac{1}{\beta(m,n)} \frac{x^{m-2}}{(1+x)^{m+n}}\,dx.$$

$$= \frac{1}{\beta(m,n)} \int_0^{1} \frac{x^{m-2}}{(1+x)^{m+n}}\,dx.$$

Let $\frac{1}{1+x} = y \Rightarrow 1+x = \frac{1}{y}$

$\Rightarrow x = \frac{1}{y} - 1 = \frac{1-y}{y}$

$\Rightarrow dx = -1/y^2\,dy$

when $x \to 0$, $y \to 1$

$x \to \infty$, $y \to 0$

Then $E\left(\frac{1}{x}\right) = \frac{1}{\beta(m,n)} \int_1^0 \left(\frac{1-y}{y}\right)^{m-2} y^{m+n}\left(-\frac{1}{y^2}\right)dy$

$$= \frac{1}{\beta(m,n)} \int_0^1 \frac{(1-y)^{m-2}}{y^{m-2}}\, y^m\, y^{n-2}\,dy$$

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$$= \frac{1}{\beta(m,n)} \int_0^1 \frac{(1-y)^{m-2}}{\cancel{y^m}\, y^{-2}} \cancel{y^m}\, y^{n-2}\, dy$$

$$= \frac{1}{\beta(m,n)} \int_0^1 y^n (1-y)^{m-2}\, dy$$

$$= \frac{1}{\beta(m,n)} \int_0^1 y^{\overline{n+1}-1} (1-y)^{\overline{m-1}-1}\, dy$$

$$= \frac{1}{\beta(m,n)} \beta(n+1, m-1)$$

$$= \frac{\cancel{\Gamma(m+n)}}{\Gamma(m)\Gamma(n)} \cdot \frac{\Gamma(n+1)\Gamma(m-1)}{\cancel{\Gamma(m+n)}}$$

$$= \frac{n\cancel{\Gamma(n)}\,\cancel{\Gamma(m-1)}}{(m-1)\cancel{\Gamma(m-1)}\,\cancel{\Gamma(n)}} = \frac{n}{m-1}$$

Now equation ① becomes

$$\boxed{H.M. = \frac{m-1}{n}}$$

Mode.

Here $f(x) = \dfrac{x^{m-1}}{\beta(m,n)(1+x)^{m+n}}$; $0 \le x < \infty$

Diff. w.r.t. x, we get

$$f'(x) = \frac{1}{\beta(m,n)} \frac{d}{dx}\left[x^{m-1} (1+x)^{-m-n} \right]$$

$$= \frac{1}{\beta(m,n)} \left[x^{m-1} (-m-n)(1+x)^{-m-n-1} + (m-1)\, x^{m-2} (1+x)^{-m-n} \right]$$

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$$= \frac{x^{m-1}(1+x)^{-m-n}}{\beta(m,n)}\left[-(m+n)(1+x)^{-1}+(m-1)x^{-1}\right]$$

$$= \frac{1}{\beta(m,n)}\frac{x^{m-1}}{(1+x)^{m+n}}\left[\frac{m-1}{x}-\frac{m+n}{1+x}\right]$$

Put $f'(x)=0$

Then $\frac{1}{\beta(m,n)}\frac{x^{m-1}}{(1+x)^{m+n}}\left[\frac{m-1}{x}-\frac{m+n}{1+x}\right]=0$

But $\frac{1}{\beta(m,n)}\frac{x^{m-1}}{(1+x)^{m+n}} \neq 0$

So $\frac{m-1}{x}-\frac{m+n}{1+x}=0$

$\Rightarrow \frac{m-1}{x}=\frac{m+n}{1+x}$

$\Rightarrow (m-1)(1+x)=(m+n)x$

$\Rightarrow m+mx-1-x=(m+n)x$

$\Rightarrow m-1+(m-1)x=(m+n)x$

$\Rightarrow (m-1)x-(m+n)x=1-m$

$\Rightarrow (\cancel{m}-1-\cancel{m}-n)x=1-m$

$\Rightarrow -(n+1)x=1-m$

$\Rightarrow \boxed{x=\frac{m-1}{n+1}}$

Diff. again w.r.t. x, we get

$$f''(x)=\frac{1}{\beta(m,n)}\left[-(m+n)\left\{x^{m-1}(-m-n-1)(1+x)^{-m-n-2}+(m-1)x^{m-2}(1+x)^{-m-n-1}\right\}\right.$$
$$\left.+(m-1)\left\{x^{m-2}(-m-n)(1+x)^{-m-n-1}+(m-2)x^{m-3}(1+x)^{-m-n}\right\}\right]$$

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$$= \frac{1}{\beta(m,n)}\Big[(m+n)(m+n+1)\, x^{m-1}(1+x)^{-m-n-2} - 2(m+n)(m-1)\, x^{m-2}(1+x)^{-m-n-1} + (m-1)(m-2)\, x^{m-3}(1+x)^{-m-n}\Big]$$

$$= \frac{1}{\beta(m,n)}\, x^{m-1}(1+x)^{-m-n}\Big[(m+n)(m+n+1)(1+x)^{-2} - 2(m+n)(m-1)\, x^{-1}(1+x)^{-1} + (m-1)(m-2)\, x^{-2}\Big]$$

$$= \frac{1}{\beta(m,n)}\, \frac{x^{m-1}}{(1+x)^{m+n}}\left[\frac{(m+n)(m+n+1)}{(1+x)^2} - \frac{2(m+n)(m-1)}{x(1+x)} + \frac{(m-1)(m-2)}{x^2}\right] \longrightarrow (1)$$

Now $\quad 1+x = 1 + \dfrac{m-1}{n+1} = \dfrac{n+1+m-1}{n+1}$

$$\Rightarrow 1+x = \frac{m+n}{n+1}$$

$$x(1+x) = \frac{(m-1)}{n+1}\left(\frac{m+n}{n+1}\right)$$

$$= \frac{m^2+mn-m-n}{(n+1)^2}$$

Consider

$$\frac{(m+n)(m+n+1)}{(1+x)^2} - \frac{2(m+n)(m-1)}{x(1+x)} + \frac{(m-1)(m-2)}{x^2}$$

$$= \frac{\cancel{(m+n)}(m+n+1)}{(m+n)^{\cancel{2}}}(n+1)^2 - \frac{2\cancel{(m+n)}\cancel{(m-1)}}{\cancel{(m-1)}\cancel{(m+n)}}(n+1)^2 + \frac{\cancel{(m-1)}(m-2)}{(m-1)^{\cancel{2}}}\,(n+1)^2$$

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$$= \frac{m+n+1}{m+n}(n+1)^2 - 2(n+1)^2 + \frac{(m-2)}{m-1}(n+1)^2$$

$$= (n+1)^2\left[\frac{m+n+1}{m+n} - 2 + \frac{m-2}{m-1}\right]$$

$$= (n+1)^2\left[\frac{(m+n+1)(m-1) - 2(m+n)(m-1) + (m-2)(m+n)}{(m+n)(m-1)}\right]$$

$$= \frac{(n+1)^2}{(m+n)(m-1)}\left[m^2 - \cancel{m} + mn - n + \cancel{m} - 1 - 2(m^2 - m + mn - n) + m^2 + mn - 2m - 2n\right]$$

$$= \frac{(n+1)^2}{(m+n)(m-1)}\left(\cancel{2m^2} - \cancel{2m} + \cancel{2mn} - 3n - 1 - \cancel{2m^2} + \cancel{2m} - \cancel{2mn} + 2n\right)$$

$$= \frac{(n+1)^2}{(m+n)(m-1)}(-n-1)$$

$$= -\frac{(n+1)^3}{(m+n)(m-1)}$$

Now eq ① becomes

$$f''(x)\Big|_{x=\frac{m-1}{n+1}} = \frac{1}{\beta(m,n)}\left(\frac{m-1}{n+1}\right)^{m-1}\left(\frac{n+1}{m+n}\right)^{m+n}\left[\frac{-(n+1)^3}{(m+n)(m-1)}\right]$$

$$= \frac{-1}{\beta(m,n)}\frac{(m-1)^{m-1}}{(n+1)^{m-1}} \cdot \frac{(n+1)^{m+n}}{(m+n)^{m+n}} \cdot \frac{(n+1)^3}{(m+n)(m-1)}$$

$$= \frac{-1}{\beta(m,n)}\frac{(m-1)^{m-2}(n+1)^{n+4}}{(m+n)^{m+n+1}} < 0$$

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Hence $x = \frac{m-1}{n+1}$ is mode.

Question 1

Show that $\beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$

Sol.

We know that,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Let

$$x = \frac{1}{1+u} \Rightarrow dx = \frac{-1}{(1+u)^2}\,du$$

$$\Rightarrow 1-x = 1 - \frac{1}{1+u}$$

$$= \frac{1+u-1}{1+u} = \frac{u}{1+u}$$

$$\boxed{\begin{aligned} u+1 &= \frac{1}{x} \\ u &= \frac{1}{x}-1 \end{aligned}}$$

when $x \to 0$, $u \to \infty$

$x \to 1$, $u \to 0$

Then,

$$\beta(m,n) = \int_\infty^0 \left(\frac{1}{1+u}\right)^{m-1}\left(\frac{u}{1+u}\right)^{n-1}\left(\frac{-1}{(1+u)^2}\right)du$$

$$= \int_0^\infty \frac{1}{(1+u)^{m-1}}\,\frac{u^{n-1}}{(1+u)^{n-1}}\,\frac{du}{(1+u)^2}$$

$$= \int_0^\infty \frac{u^{n-1}}{(1+u)^{m+n}}\,du$$

$$\Rightarrow \beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

dummy variable

as req

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A/2013 Show that for each β_1, there corresponds a pair of β_2 variate.

Question2.

Prove that $\beta(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$

Proof.

As we know that

$$\beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx + \int_1^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx + I_1 \quad \text{———} \quad (1)$$

where

$$I_1 = \int_1^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

Let $x = \frac{1}{u} \Rightarrow dx = \frac{-1}{u^2} du$

When $x \to 1$, $u \to 1$ $\quad 1+x = 1+\frac{1}{u}$

$x \to \infty$, $u \to 0$

Then $I_1 = \int_1^0 \left(\frac{1}{u}\right)^{n-1} \frac{1}{\left(\frac{u+1}{u}\right)^{m+n}} \left(-\frac{1}{u^2}\right) du$

$$= \int_0^1 \frac{1}{u^{n-1}} \frac{u^{m+n}}{(1+u)^{m+n}} \frac{du}{u^2}$$

$$= \int_0^1 u^{m+\cancel{n}-\cancel{n}+1-2} \frac{du}{(1+u)^{m+n}}$$

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$$= \int_0^1 \frac{u^{m-1}}{(1+u)^{m+n}}\, du$$

$$\Rightarrow I_1 = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$ {dummy variable}

thus eq ① becomes

$$\beta(m,n) = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\, dx + \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

$$= \int_0^1 \frac{x^{n-1} + x^{m-1}}{(1+x)^{m+n}}\, dx$$ as required.

Question 3.

Show that $\beta(\frac{1}{2}, \frac{1}{2}) = \pi$.

Sol.

We know that,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

so $$\beta(\tfrac{1}{2}, \tfrac{1}{2}) = \int_0^1 x^{-1/2}(1-x)^{-1/2}\, dx$$

$$= \int_0^1 \frac{dx}{\sqrt{x}\sqrt{1-x}}$$

$$= \int_0^1 \frac{dx}{\sqrt{x(1-x)}}$$

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Put $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$

$\sqrt{x} = \sin\theta$

$\theta = \sin^{-1}(\sqrt{x})$

when $x \to 0$, $\theta \to 0$

$x \to 1$, $\theta \to \pi/2$

Then

$$\beta(\tfrac{1}{2}, \tfrac{1}{2}) = \int_0^{\pi/2} \frac{2\sin\theta\cos\theta\, d\theta}{\sqrt{\sin^2\theta(1-\sin^2\theta)}}$$

$$= \int_0^{\pi/2} \frac{2\cancel{\sin\theta\cos\theta}}{\cancel{\sin\theta\cos\theta}}\, d\theta$$

$$= 2\int_0^{\pi/2} d\theta = 2|\theta|_0^{\pi/2} = 2(\tfrac{\pi}{2} - 0)$$

$$\Rightarrow \beta(\tfrac{1}{2}, \tfrac{1}{2}) = \pi.$$

Question 4.

Prove that $\int_0^a x^{m-1}(a-x)^{n-1}\, dx = a^{m+n-1}\beta(m,n)$

Sol.

As we know that

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx.$$

Put $x = \frac{u}{a} \Rightarrow dx = \frac{1}{a}du$

$\Rightarrow u = ax$

when $x \to 0$, $u \to 0$

$x \to 1$, $u \to a$

Then $$\beta(m,n) = \int_0^a \left(\frac{u}{a}\right)^{m-1}\left(1-\frac{u}{a}\right)^{n-1}\frac{du}{a}$$

$$= \int_0^a \frac{u^{m-1}}{a^{m-1}}\frac{(a-u)^{n-1}}{a^{n-1}}\frac{du}{a}$$

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$$= \int_0^1 \frac{u^{m-1}(a-u)^{n-1}}{a^{m+n-1}} du$$

$$= \frac{1}{a^{m+n-1}} \int_0^1 x^{m-1}(a-x)^{n-1} dx$$ {dummy variable}

$$\Rightarrow \int_0^1 x^{m-1}(a-x)^{n-1} dx = a^{m+n-1} \beta(m,n).$$

Question 5. A/2018,

V.V.V Imp Prove the relationship b/w beta and gamma functions as

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

and deduce that

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$

Sol.

As we know that

$$\Gamma(m) = \int_0^\infty e^{-x} x^{m-1} dx$$

and

$$\Gamma(n) = \int_0^\infty e^{-y} y^{n-1} dy$$

So $$\Gamma(m)\Gamma(n) = \int_0^\infty \int_0^\infty e^{-(x+y)} x^{m-1} y^{n-1} dx dy \rightarrow (1)$$

To solve eq (1), we will use the transformation as follow:

$$x = uv \quad , y = u(1-v)$$

$$x+y = \cancel{uv} + u - \cancel{uv}$$

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$$\Rightarrow x+y = u \qquad \text{———} \quad (2)$$

and $\dfrac{x}{y} = \dfrac{uv}{u(1-v)} = \dfrac{v}{1-v}$

$$\Rightarrow \frac{1-v}{v} = \frac{y}{x} \qquad \Rightarrow \frac{1}{v} - 1 = \frac{y}{x}$$

$$\Rightarrow \frac{1}{v} = 1 + y/x \qquad \Rightarrow \frac{x+y}{x} = \frac{1}{v}$$

$$\Rightarrow v = \frac{x}{x+y} = \frac{1}{1+y/x} \qquad \text{———} \quad (3)$$

As $\quad 0 \le x, y < \infty$

so from (2) and (3), we have

$0 \le u < \infty \quad , \quad 0 \le v \le 1.$

$$dx\,dy = |J|\,du\,dv \qquad \text{———} \quad (4)$$

where

J is Jacobian which is given by

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -uv - u(1-v)$$

$$= -\cancel{uv} - u + \cancel{uv} = -u$$

Thus eq (4) becomes

$$dx\,dy = |-u|\,du\,dv = u\,du\,dv$$

$$(1) \Rightarrow \Gamma(m)\Gamma(n) = \int_{v=0}^{\infty}\int_{u=0}^{\infty} e^{-u}(uv)^{m-1}(u(1-v))^{n-1}\,u\,du\,dv$$

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$$= \int_{u=0}^{\infty}\int_{v=0}^{1} e^{-u} u^{m-1} v^{m-1} u^{n-1} (1-v)^{n-1} \, u \, du \, dv$$

$$= \int_{u=0}^{\infty} e^{-u} u^{m-1} u^{n-1} u \, du \int_{v=0}^{1} v^{m-1} (1-v)^{n-1} \, dv$$

$$= \int_{u=0}^{\infty} e^{-u} u^{m+n-1} \, du \int_{v=0}^{1} v^{m-1} (1-v)^{n-1} \, dv$$

$$= \Gamma(m+n) \cdot \beta(m,n)$$

$$\Rightarrow \beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$$

$$\Rightarrow \beta\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \frac{\left(\Gamma\left(\frac{1}{2}\right)\right)^2}{\Gamma(1)}$$

Since, we know that

$$\Gamma(1) = 1 \quad \text{and} \quad \beta\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \pi$$

Therefore

$$\left(\Gamma\left(\tfrac{1}{2}\right)\right)^2 = \pi$$

$$\Rightarrow \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$ which is required.

Question 6. V.V.V. Imp

If X and Y are independent Gamma variates with parameters μ and ν resp. then find the distribution of the following variables and show that

(i) $U = X+Y$ is a Gamma variate with parameter $\mu+\nu$.

(ii) $V = \dfrac{X}{X+Y}$ is a $\beta_1(\mu,\nu)$ variate,

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(iii) $W = \frac{X}{Y}$ is a $\beta_2(\mu, \nu)$ variate.

(iv) $U = X + Y$ and $V = \frac{X}{X+Y}$ are independent

(v) $U = X + Y$ and $W = X/Y$ are independent

(vi) $U = X + Y$ and $Z = \frac{X-Y}{X+Y}$ are independent.

Solution.

Since X and Y are independent Gamma variates with parameters μ and ν resp. Therefore

$$f_1(x)\,dx = \frac{1}{\Gamma(\mu)} e^{-x} x^{\mu-1} dx \quad ; 0 \le x < \infty ; \mu > 0$$

$$f_2(y)\,dy = \frac{1}{\Gamma(\nu)} e^{-y} y^{\nu-1} dy \quad ; 0 \le y < \infty ; \nu > 0$$

and their joint distribution is given by

$$dP = f(x, y)\,dx\,dy$$

$$= f_1(x) \cdot f_2(y)\,dx\,dy$$

$$= \frac{1}{\Gamma(\mu)\Gamma(\nu)} e^{-(x+y)} x^{\mu-1} y^{\nu-1} dx\,dy \rightarrow (1)$$

$$; 0 \le x, y < \infty$$

(i)

Probability distribution of $U = X + Y$ i.e. $u = x + y$ is given by

$$P(X+Y \le u) = \iint_{x+y \le u} \frac{1}{\Gamma(\mu)\Gamma(\nu)} e^{-(x+y)} x^{\mu-1} y^{\nu-1} dx\,dy$$

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Since $u = x + y$

$\Rightarrow x = u - y$ $\qquad \Rightarrow dx = du$

$= u\left(1 - \frac{y}{u}\right)$

Therefore,

$$P(X+Y \le u) = \int_{u=0}^{\infty}\int_{y=0}^{u} \frac{1}{\Gamma(\mu)\Gamma(\nu)} e^{-u} u^{\mu-1}\left(1-\frac{y}{u}\right)^{\mu-1} y^{\nu-1}\, du\, dy$$

$$= \int_{u=0}^{\infty} \frac{1}{\Gamma(\mu)\Gamma(\nu)} e^{-u} u^{\mu-1} du \int_{y=0}^{u} \left(1-\frac{y}{u}\right)^{\mu-1} y^{\nu-1} dy$$

Put $\frac{y}{u} = t \Rightarrow y = ut \Rightarrow dy = u\,dt$

when $y \to 0$, $t \to 0$

$y \to u$, $t \to 1$

Then

$$P(X+Y \le u) = \int_{u=0}^{\infty} \frac{1}{\Gamma(\mu)\Gamma(\nu)} e^{-u} u^{\mu-1} du \int_{t=0}^{1} (1-t)^{\mu-1} (ut)^{\nu-1} u\, dt$$

$$= \int_{u=0}^{\infty} \frac{1}{\Gamma(\mu)\Gamma(\nu)} e^{-u} u^{\mu-1} du \int_{t=0}^{1} (1-t)^{\mu-1} t^{\nu-1} u^{\nu} dt$$

$$= \int_{u=0}^{\infty} \frac{1}{\Gamma(\mu)\Gamma(\nu)} e^{-u} u^{\mu-1} u^{\nu} du \int_{t=0}^{1} (1-t)^{\mu-1} t^{\nu-1} dt$$

$$= \int_{u=0}^{\infty} \frac{1}{\Gamma(\mu)\Gamma(\nu)} e^{-u} u^{\mu+\nu-1} du \int_{t=0}^{1} t^{\nu-1} (1-t)^{\mu-1} dt$$

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$$= \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_{u=0}^{\infty} e^{-u} u^{\mu+\nu-1} du . \beta(\mu,\nu)$$

$$= \frac{\beta(\mu,\nu)}{\Gamma(\mu)\Gamma(\nu)} \int_{u=0}^{\infty} e^{-u} u^{\mu+\nu-1} du$$

But

$$\beta(\mu,\nu) = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}$$

$$\Rightarrow \frac{1}{\Gamma(\mu+\nu)} = \frac{\beta(\mu,\nu)}{\Gamma(\mu)\Gamma(\nu)}$$

So

$$P(X+Y \le u) = \frac{1}{\Gamma(\mu+\nu)} \int_{0}^{\infty} e^{-u} u^{\mu+\nu-1} du$$

which shows that $U = X+Y$ is a Gamma variate with parameter $\mu+\nu$.

(ii) A/2017,

Probability distribution of $V = \frac{X}{X+Y}$.

i.e. $V = \frac{x}{x+y}$ is given by

$$P\left(\frac{X}{X+Y} \le v\right) = \iint_{\frac{x}{x+y} \le v} \frac{1}{\Gamma(\mu)\Gamma(\nu)} e^{-(x+y)} x^{\mu-1} y^{\nu-1} dx dy$$

Put $x = r\cos^2\theta$, $y = r\sin^2\theta$

Then, the transformation is

$$dx dy = |J| \, dr d\theta$$

where J is the Jacobian, of the transformation, which is given by

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$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos^2\theta & -2r\cos\theta\sin\theta \\ \sin^2\theta & 2r\sin\theta\cos\theta \end{vmatrix}$$

$$= 2r\sin\theta\cos^3\theta + 2r\cos\theta\sin^3\theta$$
$$= 2r\sin\theta\cos\theta\,(\cos^2\theta + \sin^2\theta)$$
$$= 2r\sin\theta\cos\theta.$$

Thus,

$$dx\,dy = 2r\sin\theta\cos\theta\,dr\,d\theta.$$

$$x + y = r(\cos^2\theta + \sin^2\theta)$$

$$\Rightarrow x + y = r$$

and $\tan^2\theta = \frac{y}{x} \quad \Rightarrow \tan\theta = \sqrt{\frac{y}{x}}$

$$\Rightarrow \theta = \tan^{-1}\sqrt{y/x}$$

$$\Rightarrow \quad 0 \le r < \infty \quad ; \quad 0 \le \theta \le \frac{\pi}{2}$$

Thus,

$$P\left(\frac{X}{X+Y} \le v\right) = \int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} \frac{e^{-r}}{\Gamma(\mu)\Gamma(\nu)} (r\cos^2\theta)^{\mu-1} (r\sin^2\theta)^{\nu-1}\, 2r\sin\theta\cos\theta\, dr\, d\theta$$

$$= \frac{2}{\Gamma(\mu)\Gamma(\nu)} \int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r}\, r^{\mu-1} \cos^{2\mu-2}\theta\, r^{\nu-1} \sin^{2\nu-2}\theta\, r\sin\theta\cos\theta\, dr\, d\theta$$

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$$= \frac{2}{\Gamma(\mu)\Gamma(\nu)} \int_{r=0}^{\infty} \int_{\theta=0}^{\pi/2} e^{-r} r^{\mu+\nu-1} \cos^{2\mu-1}\theta \sin^{2\nu-1}\theta \, dr \, d\theta$$

$$= \frac{2}{\Gamma(\mu)\Gamma(\nu)} \int_{r=0}^{\infty} e^{-r} r^{\mu+\nu-1} dr \int_{\theta=0}^{\pi/2} \cos^{2\mu-1}\theta \sin^{2\nu-1}\theta \, d\theta.$$

Put $\cos^2\theta = t \Rightarrow dt = -2\cos\theta\sin\theta \, d\theta.$

when $\theta \to 0$; $t \to 1$

$\theta \to \pi/2$, $t \to 0$

Then

$$P\left(\frac{X}{X+Y} \le v\right) = \frac{\cancel{2}}{\Gamma(\mu)\Gamma(\nu)} \int_{r=0}^{\infty} e^{-r} r^{\mu+\nu-1} dr \int_{t=1}^{0} t^{\mu-1}(1-t)^{\nu-1} \left(-\frac{dt}{\cancel{2}}\right)$$

$$= \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_0^{\infty} e^{-r} r^{\mu+\nu-1} dr \int_0^1 t^{\mu-1}(1-t)^{\nu-1} dt$$

$$= \frac{1}{\Gamma(\mu)\Gamma(\nu)} \Gamma(\mu+\nu) \int_0^1 t^{\mu-1}(1-t)^{\nu-1} dt$$

$$= \frac{1}{\beta(\mu,\nu)} \int_0^1 t^{\mu-1}(1-t)^{\nu-1} dt$$

$$= \int_0^1 \frac{1}{\beta(\mu,\nu)} t^{\mu-1}(1-t)^{\nu-1} dt.$$

which shows that $V = \frac{X}{X+Y}$ is a $\beta_1(\mu,\nu)$ variate.

(iii) S/2018 , A/2016

Probabability distribution of $W = \frac{X}{Y}$ i.e. $w = \frac{x}{y}$ is given by

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$$P\left(\frac{X}{Y} \le w\right) = \iint_{\frac{x}{y} \le w} \frac{1}{\Gamma(\mu)\Gamma(\nu)} e^{-(x+y)} x^{\mu-1} y^{\nu-1} \, dx\,dy$$

Since

$$w = \frac{x}{y} \Rightarrow x = yw \Rightarrow dx = y\,dw$$

Also $x + y = yw + y$

$= y(1+w)$

Therefore,

$$P\left(\frac{X}{Y} \le w\right) = \int_{w=0}^{\infty}\int_{y=0}^{\infty} \frac{e^{-y(1+w)}}{\Gamma(\mu)\Gamma(\nu)} (yw)^{\mu-1} y^{\nu-1} y\,dw\,dy$$

$$= \int_{w=0}^{\infty}\int_{y=0}^{\infty} \frac{e^{-y(1+w)}}{\Gamma(\mu)\Gamma(\nu)} y^{\mu-1} w^{\mu-1} y^{\nu} \, dw\,dy$$

$$= \int_{w=0}^{\infty} \frac{w^{\mu-1}}{\Gamma(\mu)\Gamma(\nu)} dw \int_{y=0}^{\infty} e^{-y(1+w)} y^{\mu+\nu-1} dy$$

$$= \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_{w=0}^{\infty} w^{\mu-1} dw \int_{y=0}^{\infty} e^{-y(1+w)} y^{\mu+\nu-1} dy$$

Put $y(1+w) = t$

$\Rightarrow (1+w)\,dy = dt$

$y \to 0, \quad t \to 0$

$y \to \infty, \quad t \to \infty$

Then

$$P\left(\frac{X}{Y} \le w\right) = \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_0^{\infty} w^{\mu-1} dw \int_0^{\infty} e^{-t} \left(\frac{t}{1+w}\right)^{\mu+\nu-1} \frac{dt}{1+w}$$

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$$= \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_0^\infty w^{\mu-1} dw \int_0^\infty e^{-t} \frac{t^{\mu+\nu-1}}{(1+w)^{\mu+\nu}} dt$$

$$= \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_0^\infty \frac{w^{\mu-1}}{(1+w)^{\mu+\nu}} dw \int_0^\infty e^{-t} t^{\mu+\nu-1} dt$$

$$= \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_0^\infty \frac{w^{\mu-1}}{(1+w)^{\mu+\nu}} dw \cdot \Gamma(\mu+\nu)$$

$$= \frac{\Gamma(\mu+\nu)}{\Gamma(\mu)\Gamma(\nu)} \int_0^\infty \frac{w^{\mu-1}}{(1+w)^{\mu+\nu}} dw$$

$$= \frac{1}{\beta(\mu,\nu)} \int_0^\infty \frac{w^{\mu-1}}{(1+w)^{\mu+\nu}} dw$$

$$= \int_0^\infty \frac{1}{\beta(\mu,\nu)} \frac{w^{\mu-1}}{(1+w)^{\mu+\nu}} dw$$

which shows that $W = X/Y$ is $\beta_2(\mu,\nu)$ variate

(iv)

To show $U = X+Y$ and $V = \frac{X}{X+Y}$ are independent, we have

$$u = x+y \quad \text{and} \quad v = \frac{x}{x+y}$$

$$\Rightarrow \quad v = \frac{x}{u} \quad \Rightarrow \quad \boxed{x = uv}$$

and

$$y = u - x \quad \Rightarrow y = u - uv$$

$$\Rightarrow \quad \boxed{y = u(1-v)}$$

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Jacobian of transformation is

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix}$$

$$= \cancel{-uv} - u + \cancel{uv}$$

$$\Rightarrow J = -u$$

where

$0 \leq x, y < \infty$, $0 \leq u < \infty$ and $0 \leq v < 1$.

The joint distribution of U and V is

$$dG(u,v) = g(u,v)\, du\, dv$$

From ①, we have

$$dG(u,v) = \frac{1}{\Gamma(\mu)\Gamma(\nu)} e^{-u} (uv)^{\mu-1} \{u(1-v)\}^{\nu-1} u\, du\, dv$$

where $dx\,dy = |J|\, du\, dv$

$$= \frac{1}{\Gamma(\mu)\Gamma(\nu)} e^{-u} u^{\mu-1} v^{\mu-1} u^{\nu-1} (1-v)^{\nu-1} u\, du\, dv$$

$$= \frac{1}{\Gamma(\mu)\Gamma(\nu)} e^{-u} u^{\mu+\nu-1} v^{\mu-1} (1-v)^{\nu-1} du\, dv$$

$$= \frac{1}{\Gamma(\mu)\Gamma(\nu)} e^{-u} u^{\mu+\nu-1} du\, v^{\mu-1} (1-v)^{\nu-1} dv$$

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$$= \frac{1}{\beta(\mu,\nu)\,\Gamma(\mu+\nu)} e^{-u} u^{\mu+\nu-1} du \; v^{\mu-1} (1-v)^{\nu-1} dv$$

$$= \left\{ \frac{1}{\Gamma(\mu+\nu)} e^{-u} u^{\mu+\nu-1} du \right\} \left\{ \frac{1}{\beta(\mu,\nu)} v^{\mu-1} (1-v)^{\nu-1} dv \right\}.$$

$$= g_1(u)du \; g_2(v)\,dv$$

where

$$g_1(u)\,du = \frac{1}{\Gamma(\mu+\nu)} e^{-u} u^{\mu+\nu-1} du \; ; \; 0 \le u < \infty \quad \text{—— (2)}$$

and

$$g_2(v)\,dv = \frac{1}{\beta(\mu,\nu)} v^{\mu-1} (1-v)^{\nu-1} dv \; ; \; 0 \le v < \infty \quad \text{—— (3)}$$

(2) and (3) $\Rightarrow$ U and V are independently distributed as $\gamma(\mu+\nu)$ variate and $\beta_1(\mu,\nu)$ variate resp.

(V)

To Show $U = X + Y$ and $W = \frac{X}{Y}$ are independent, we have

$$u = x + y \quad \text{and} \quad w = \frac{x}{y}$$

$$\Rightarrow x = yw$$

$$\Rightarrow u = yw + y = y(w+1)$$

$$\Rightarrow \boxed{y = \frac{u}{1+w}}$$

and $x = u - y = u - \frac{u}{1+w}$

$$\Rightarrow x = u\left(1 - \frac{1}{1+w}\right)$$

$$= u\left(\frac{1+w-1}{1+w}\right) \Rightarrow \boxed{x = \frac{uw}{1+w}}$$

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Jacobian of transformation is

$$J = \frac{\partial(x,y)}{\partial(u,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial w} \end{vmatrix}$$

$$\because \frac{\partial x}{\partial w} = u\left\{\frac{1+w-w}{(1+w)^2}\right\} = \frac{u}{(1+w)^2}$$

$$= \begin{vmatrix} \frac{w}{1+w} & \frac{u}{(1+w)^2} \\ \frac{1}{1+w} & \frac{-u}{(1+w)^2} \end{vmatrix}$$

$$= \frac{-uw}{(1+w)^3} - \frac{u}{(1+w)^3}$$

$$= \frac{-uw - u}{(1+w)^3} = \frac{-u(1+w)}{(1+w)^3}$$

$$= \frac{-u}{(1+w)^2}$$

where

$0 \le x, y < \infty$, $0 \le u < \infty$ and $0 \le w < \infty$

The joint distribution of U and W is

$$dG(u,w) = g(u,w)\,du\,dw$$

From ①, we have

$$dG(u,w) = \frac{1}{\Gamma(\mu)\Gamma(\nu)} e^{-u} \left(\frac{uw}{1+w}\right)^{\mu-1} \left(\frac{u}{1+w}\right)^{\nu-1} |J|\,du\,dw$$

where $dx\,dy = |J|\,du\,dw$

$$= \frac{1}{\Gamma(\mu)\Gamma(\nu)} e^{-u} \frac{u^{\mu-1} w^{\mu-1}}{(1+w)^{\mu-1}} \cdot \frac{u^{\nu-1}}{(1+w)^{\nu-1}} \frac{u}{(1+w)^2}\,du\,dw$$

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$$= \frac{1}{\Gamma(\mu)\Gamma(\nu)} e^{-u} \frac{u^{\mu+\nu-1} \quad w^{\mu-1}}{(1+w)^{\mu+\nu}} du\,dw$$

$$= \frac{1}{\Gamma(\mu)\Gamma(\nu)} e^{-u} u^{\mu+\nu-1} du \frac{w^{\mu-1}}{(1+w)^{\mu+\nu}} dw$$

$$= \frac{1}{\beta(\mu,\nu)\Gamma(\mu+\nu)} e^{-u} u^{\mu+\nu-1} du \frac{w^{\mu-1}}{(1+w)^{\mu+\nu}} dw$$

$$= \left\{ \frac{1}{\Gamma(\mu+\nu)} e^{-u} u^{\mu+\nu-1} du \right\} \left\{ \frac{1}{\beta(\mu,\nu)} \frac{w^{\mu-1}}{(1+w)^{\mu+\nu}} dw \right\}$$

$$= g_1(u)\,du \cdot g_2(w)\,dw$$

where

$$g_1(u)\,du = \frac{1}{\Gamma(\mu+\nu)} e^{-u} u^{\mu+\nu-1} du \quad \text{——— (2)}$$

and

$$g_2(w)\,dw = \frac{1}{\beta(\mu,\nu)} \frac{w^{\mu-1}}{(1+w)^{\mu+\nu}} dw \quad \text{——— (3)}$$

(2) and (3) $\Rightarrow$ U and W are independently distributed as $\gamma(\mu+\nu)$ variate and $\beta_2(\mu,\nu)$ variate respectively.

(vi)

To show $U = X+Y$ and $Z = \frac{X-Y}{X+Y}$ are independent, we have

$$u = x+y \quad \text{and} \quad z = \frac{x-y}{x+y}$$

$$\Rightarrow z = \frac{x-y}{u} \Rightarrow x-y = uz$$

Adding: $2x = u+uz \Rightarrow u(1+z) = 2x$

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$$\Rightarrow \boxed{x = \frac{u}{2}(1+z)}$$

<u>Subtracting</u>: $2y = u - uz = u(1-z)$

$$\Rightarrow \boxed{y = \frac{u}{2}(1-z)}$$

Jacobian of transformation is

$$J = \frac{\partial(x,y)}{\partial(u,z)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1+z}{2} & \frac{u}{2} \\ \frac{1-z}{2} & \frac{-u}{2} \end{vmatrix}$$

$$= -\frac{u}{4}(1+z) - \frac{u}{4}(1-z)$$

$$= -\frac{u}{4}(1+\cancel{z}+1-\cancel{z}) = -\frac{2u}{4}$$

$$\Rightarrow J = -\frac{u}{2}$$

where $0 \le x, y < \infty$, $0 \le u < \infty$

and $-1 \le z \le 1$

Hence, from ①, the joint distribution of U and Z is

$$dG(u,z) = g(u,z)\,du\,dz$$

$$= \frac{1}{\Gamma(\mu)\Gamma(\nu)} e^{-u} \left(\frac{u}{2}(1+z)\right)^{\mu-1} \left(\frac{u}{2}(1-z)\right)^{\nu-1} |J|\,du\,dz$$

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where $dxdy = |J|\, du\, dz$

$$\Rightarrow dG(u,z) = \frac{1}{\Gamma(\mu)\Gamma(\nu)} e^{-u} \left(\frac{u}{2}\right)^{\mu-1} (1+z)^{\mu-1} \left(\frac{u}{2}\right)^{\nu-1} (1-z)^{\nu-1} \frac{u}{2}\, du\, dz$$

$$= \frac{1}{\Gamma(\mu)\Gamma(\nu)} e^{-u} \frac{u^{\mu-1}}{2^{\mu-1}} (1+z)^{\mu-1} \frac{u^{\nu-1}}{2^{\nu-1}} (1-z)^{\nu-1} \frac{u}{2}\, du\, dz$$

$$= \frac{1}{\Gamma(\mu+\nu)\beta(\mu,\nu)} e^{-u} u^{\mu+\nu-1} \frac{(1+z)^{\mu-1}(1-z)^{\nu-1}}{2^{\mu+\nu-1}}\, du\, dz$$

$$= \left\{ \frac{1}{\Gamma(\mu+\nu)} e^{-u} u^{\mu+\nu-1} du \right\} \left\{ \frac{1}{\beta(\mu,\nu)} \frac{(1+z)^{\mu-1}(1-z)^{\nu-1}}{2^{\mu+\nu-1}} dz \right\}$$

$$= g_1(u)\, du \cdot g_2(z)\, dz$$

where

$$g_1(u)\, du = \frac{1}{\Gamma(\mu+\nu)} e^{-u} u^{\mu+\nu-1} du \quad \text{——— (2)}$$

and

$$g_2(z)\, dz = \frac{1}{\beta(\mu,\nu)} \frac{(1+z)^{\mu-1}(1-z)^{\nu-1}}{2^{\mu+\nu-1}} dz \quad \text{——— (3)}$$

(2) and (3) $\Rightarrow$ U and Z are independently distributed.

Question 7.

If X and Y be resp. independent $\beta_1(\mu,\nu)$ variate and $\gamma(\mu+\nu)$ variate, then find a p.d.f. for XY and show that it is $\gamma(\mu)$ variate.

Sol.

Since X and Y are independent $\beta_1(\mu,\nu)$

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and $\gamma(\mu+\nu)$ variate respectively.

Therefore,

$$f_1(x)dx = \frac{1}{\beta(\mu,\nu)} x^{\mu-1}(1-x)^{\nu-1} dx; \; 0 \le x \le 1$$

and $$f_2(y)\,dy = \frac{1}{\Gamma(\mu+\nu)} e^{-y} y^{\mu+\nu-1} dy; \; 0 \le y < \infty$$

and their joint distribution is given by

$$dP = f(x,y)\,dx\,dy$$
$$= f_1(x)\,dx \cdot f_2(y)\,dy$$
$$= f_1(x) \cdot f_2(y)\,dx\,dy$$

$$= \frac{1}{\beta(\mu,\nu)\Gamma(\mu+\nu)} e^{-y} y^{\mu+\nu-1} x^{\mu-1}(1-x)^{\nu-1} dx\,dy \quad ; 0 \le x \le 1, \; 0 \le y < \infty.$$

Now, probability distribution of $U = XY$ i.e. $u = xy$ is given by

$$P(XY \le u) = \iint_{xy \le u} \frac{1}{\beta(\mu,\nu)\Gamma(\mu+\nu)} e^{-y} y^{\mu+\nu-1} x^{\mu-1}(1-x)^{\nu-1} dx\,dy$$

Since $u = xy \Rightarrow y = u/x \Rightarrow dy = \frac{du}{x}$

Therefore,

$$P(XY \le u) = \int_{u=0}^{\infty} \int_{x=0}^{1} \frac{e^{-\frac{u}{x}}}{\Gamma(\mu)\Gamma(\nu)} \left(\frac{u}{x}\right)^{\mu+\nu-1} x^{\mu-1}(1-x)^{\nu-1} \frac{dx\,du}{x}$$

$$= \int_{u=0}^{\infty} \int_{x=0}^{1} \frac{e^{-\frac{u}{x}}}{\Gamma(\mu)\Gamma(\nu)} \frac{u^{\mu+\nu-1}}{x^{\mu+\nu-1}} x^{\mu-1}(1-x)^{\nu-1} \frac{dx\,du}{x}$$

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$$= \int_{u=0}^{\infty}\int_{x=0}^{1} \frac{e^{-u/x}}{\Gamma(\mu)\Gamma(\nu)} \frac{u^{\mu+\nu-1}}{\cancel{x^{\mu}}\, x^{\nu}} \cancel{x^{\mu}} \cdot x^{-1} (1-x)^{\nu-1}\, dx\, du$$

$$= \int_{u=0}^{\infty}\int_{x=0}^{1} \frac{e^{-u/x}\, u^{\mu+\nu-1}}{\Gamma(\mu)\Gamma(\nu)} \frac{(1-x)^{\nu-1}}{x^{\nu+1}}\, dx\, du$$

$$= \int_{u=0}^{\infty}\int_{x=0}^{1} \frac{e^{-\frac{u}{x}}\, u^{\mu+\nu-1}}{\Gamma(\mu)\Gamma(\nu)} \frac{(1-x)^{\nu-1}}{x^{\nu-1}\, x^{2}}\, dx\, du$$

$$= \int_{u=0}^{\infty}\int_{x=0}^{1} \frac{e^{-\frac{u}{x}}\, u^{\mu+\nu-1}}{\Gamma(\mu)\Gamma(\nu)} \left(\frac{1-x}{x}\right)^{\nu-1} \frac{dx\, du}{x^{2}}$$

$$= \int_{u=0}^{\infty}\int_{x=0}^{1} \frac{e^{-\frac{u}{x}}\, u^{\mu+\nu-1}}{\Gamma(\mu)\Gamma(\nu)} \left(\frac{1}{x}-1\right)^{\nu-1} \frac{dx\, du}{x^{2}}$$

Put $\frac{1}{x} - 1 = t \Rightarrow \frac{1}{x} = t+1$

$\Rightarrow x = \frac{1}{1+t} \Rightarrow dx = \frac{-dt}{(1+t)^2}$

when $x \to 0$, $t \to \infty$

$x \to 1$, $t \to 0$

Then

$$P(XY \le u) = \int_{u=0}^{\infty}\int_{t=\infty}^{0} \frac{e^{-u(1+t)}}{\Gamma(\mu)\Gamma(\nu)} u^{\mu+\nu-1}\, t^{\nu-1} \cancel{(1+t)^{2}} \left(\frac{-dt}{\cancel{(1+t)^{2}}}\right) du$$

$$= \int_{u=0}^{\infty}\int_{t=0}^{\infty} \frac{e^{-u-ut}}{\Gamma(\mu)\Gamma(\nu)} u^{\mu+\nu-1}\, t^{\nu-1}\, dt\, du$$

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$$= \int_{u=0}^{\infty}\int_{t=0}^{\infty} \frac{e^{-u} e^{-ut}}{\Gamma(\mu)\Gamma(\nu)} u^{\mu+\nu-1} t^{\nu-1} \, dt \, du$$

$$= \int_{u=0}^{\infty} \frac{e^{-u} u^{\mu+\nu-1}}{\Gamma(\mu)\Gamma(\nu)} du \int_{t=0}^{\infty} e^{-ut} t^{\nu-1} dt$$

$$= \frac{1}{\Gamma(\mu)\cancel{\Gamma(\nu)}} \int_0^{\infty} e^{-u} u^{\mu+\nu-1} du \cdot \frac{\cancel{\Gamma(\nu)}}{u^{\nu}}$$

$$= \frac{1}{\Gamma(\mu)} \int_0^{\infty} e^{-u} u^{\mu} \cdot u^{\nu-1} \frac{du}{u^{\nu}}$$

$$= \frac{1}{\Gamma(\mu)} \int_0^{\infty} e^{-u} \cancel{u^{\nu}} u^{\mu-1} \frac{du}{\cancel{u^{\nu}}}$$

$$= \frac{1}{\Gamma(\mu)} \int_0^{\infty} e^{-u} u^{\mu-1} du$$

which shows that $U = XY$ is Gamma variate with parameter μ.

Question 8.

If X is a $\beta_2(l, m)$ variate then $\frac{1}{X}$ is $\beta_2(l, m)$ variate.

Sol.

Given that X is $\beta_2(l, m)$ variate. Therefore

$$f(x) = \frac{1}{\beta(l,m)} \frac{x^{l-1}}{(1+x)^{l+m}} \quad ; 0 \le x < \infty$$

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$$\Rightarrow \int_0^\infty f(x)\,dx = \int_0^\infty \frac{1}{\beta(l,m)} \frac{x^{l-1}}{(1+x)^{l+m}}\,dx$$

Put

$$x = \frac{1}{u} \quad \Rightarrow dx = -\frac{1}{u^2}\,du$$

when $x \to 0$, $u \to \infty$

$x \to \infty$, $u \to 0$

Then $$\int_0^\infty f(x)\,dx = \frac{1}{\beta(l,m)} \int_\infty^0 \left(\frac{1}{u}\right)^{l-1} \frac{1}{\left(1+\frac{1}{u}\right)^{l+m}} \frac{-du}{u^2}$$

$$= \frac{1}{\beta(l,m)} \int_0^\infty \frac{1}{u^{l-1}} \frac{u^{l+m}}{(1+u)^{l+m}} \frac{du}{u^2}$$

$$= \frac{1}{\beta(l,m)} \int_0^\infty \frac{\cancel{u^l} \cdot u^m}{\cancel{u^l}\, u\, (1+u)^{l+m}}\,du$$

$$= \frac{1}{\beta(l,m)} \int_0^\infty \frac{u^{m-1}}{(1+u)^{l+m}}\,du.$$

$$= \frac{1}{\beta(l,m)} \int_0^\infty \frac{x^{m-1}}{(1+x)^{l+m}}\,dx$$ dummy index

$\Rightarrow \frac{1}{X}$ is also a $\beta_2(l,m)$ ~~parameter~~ variate.

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Gaussian Distribution

Normal Distribution

The normal distribution is the limiting form of the binomial distribution by increasing n, the number of trials, to a very large number for a fixed value of p.

A normal distribution is defined by the p.d.f.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \text{ for } -\infty < x < \infty \text{ and } \sigma > 0.$$

where μ is mean, σ is the standard deviation and $\pi = 3.1416$ and $e = 2.7183$ are constants.

It is denoted by $N(x; \mu, \sigma^2)$ where μ and σ are its parameters.

Standardized Normal Distribution.

A normal variate with $\mu = 0$ and $\sigma^2 = 1$ is called standardized normal variable. It is denoted by Z and is given by

$$Z = \frac{X-\mu}{\sigma}$$

$$\text{Mean}(Z) = E(Z)$$

$$= \frac{1}{\sigma} E(X-\mu) = \frac{1}{\sigma}(\mu-\mu) = 0$$

$$\Rightarrow \mu = 0$$

$$\text{Vari}(Z) = \text{Var}\left(\frac{X-\mu}{\sigma}\right)$$

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$$= E\left(\frac{x-\mu}{\sigma}\right)^2 - \left(E\left(\frac{x-\mu}{\sigma}\right)\right)^2$$

$$= \frac{1}{\sigma^2} E(x-\mu)^2 - \mu^2 \quad \text{———} \quad (1)$$

Consider

$$E(x-\mu)^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put $z = \dfrac{x-\mu}{\sigma}$ $\Rightarrow x-\mu = \sigma z$

$\Rightarrow x = \mu + \sigma z$

$\Rightarrow dx = \sigma\, dz$

when $x \longrightarrow -\infty$, $z \longrightarrow -\infty$

$x \longrightarrow +\infty$, $z \longrightarrow +\infty$.

$$E(x-\mu)^2 = \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z)^2 e^{-\frac{1}{2}z^2} \cancel{\sigma}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma^2 z^2 e^{-\frac{1}{2}z^2} dz$$

even

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{1}{2}z^2} dz = \frac{2\sigma^2}{\sqrt{2\pi}} \int_{0}^{\infty} z^2 e^{-\frac{1}{2}z^2} dz$$

Put $\frac{1}{2}z^2 = y$ $\Rightarrow z^2 = 2y$

$\Rightarrow z = \sqrt{2y}$ $\Rightarrow dz = \frac{1}{\cancel{2}}(2y)^{-1/2} \cancel{2}\, dy$

$\Rightarrow dz = \dfrac{dy}{\sqrt{2y}}$

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$$= \frac{2a^2}{\sqrt{2\pi}} \int_0^\infty 2y \cdot e^{-y} \frac{dy}{\sqrt{2y}}$$

$$= \frac{4a^2}{\sqrt{4\pi}} \int_0^\infty y^{1/2} e^{-y} \, dy$$

$$= \frac{2a^2}{\sqrt{\pi}} \int_0^\infty e^{-y} y^{\frac{3}{2}-1} \, dy$$

$$= \frac{2a^2}{\sqrt{\pi}} \cdot \Gamma\left(\frac{3}{2}\right)$$

$$= \frac{\cancel{2}a^2}{\sqrt{\pi}} \frac{1}{\cancel{2}} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{a^2}{\cancel{\sqrt{\pi}}} \cancel{\sqrt{\pi}} \qquad \because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Rightarrow E(X-\mu)^2 = a^2$$

Even: $f(-x) = f(x)$
$\int_{-\infty}^{\infty} = 2\int_0^\infty$

odd: $f(-x) = -f(x)$
$\int_{-\infty}^{\infty} = 2(0)$

Now equation ① becomes

$$V(Z) = \frac{1}{a^2} a^2 - 0 = 1$$

The normal probability distribution of Z which has zero mean and unit variance, is called standarized normal distribution or unit normal distribution and is denoted by $N(x; 0, 1)$

The p.d.f of Z is denoted by $\varphi(z)$ as

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

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Properties of Normal Distr.

① The function $f(x)$ defining the normal distribution is a proper p.d.f.

Proof.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put $z = \dfrac{x-\mu}{\sigma} \Rightarrow x-\mu = \sigma z$

$\Rightarrow dx = \sigma\, dz$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\cancel{\sigma}\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2}\, \cancel{\sigma}\, dz$$

even

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{1}{2}z^2} dz$$

Put $+\frac{1}{2}z^2 = y \Rightarrow z^2 = 2y$

$\Rightarrow z = \sqrt{2y} \Rightarrow dz = \dfrac{1}{\sqrt{2y}}\, dy$

Then

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-y}\, \frac{1}{\sqrt{2y}}\, dy$$

$$= \frac{2}{\sqrt{4\pi}} \int_{0}^{\infty} e^{-y}\, y^{-1/2}\, dy$$

$$= \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-y}\, y^{\frac{1}{2}-1}\, dy$$

$$= \frac{1}{\sqrt{\pi}}\, \Gamma\left(\tfrac{1}{2}\right) = \frac{1}{\cancel{\sqrt{\pi}}}\, \cancel{\sqrt{\pi}} \quad \because \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

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$= 2\int_{-\infty}^{\infty} f(x)\,dx = 1.$

Thus the total area under the normal curve is unity and hence the function $f(x)$ defines proper p.d.f.

② The mean and variance of the normal distribution are μ and σ^2 respectively.

Proof.

Mean. $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$

A/2014

$$= \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put $z = \frac{x-\mu}{\sigma}$ $\Rightarrow x-\mu = \sigma z$

$\Rightarrow dx = \sigma\,dz.$

$$\Rightarrow E(X) = \int_{-\infty}^{\infty} (\mu + \sigma z) \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \cancel{\sigma}\,dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma z) e^{-\frac{1}{2}z^2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} \mu e^{-\frac{1}{2}z^2} dz + \int_{-\infty}^{\infty} \sigma z e^{-\frac{1}{2}z^2} dz \right]$$

$$= \frac{\mu}{\sqrt{2\pi}} \overset{\text{even}}{\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz} + \frac{\sigma}{\sqrt{2\pi}} \overset{\text{odd}}{\int_{-\infty}^{\infty} z e^{-\frac{1}{2}z^2} dz}$$

$$= 2\frac{\mu}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{1}{2}z^2} dz + \frac{2\sigma}{\sqrt{2\pi}}(0)$$

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$$= \frac{2\mu}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}z^2} dz$$

Put $\frac{1}{2}z^2 = y \Rightarrow z^2 = 2y$

$\Rightarrow z = \sqrt{2y} \Rightarrow dz = \frac{1}{\sqrt{2y}} dy$

$$\Rightarrow E(x) = \frac{2\mu}{\sqrt{2\pi}} \int_0^\infty e^{-y} \frac{dy}{\sqrt{2y}}$$

$$= \frac{\mu}{\sqrt{\pi}} \int_0^\infty e^{-y} y^{-1/2} dy$$

$$= \frac{\mu}{\sqrt{\pi}} \int_0^\infty e^{-y} y^{\frac{1}{2}-1} dy$$

$$= \frac{\mu}{\sqrt{\pi}} \Gamma(\tfrac{1}{2}) = \frac{\mu}{\cancel{\sqrt{\pi}}} \cancel{\sqrt{\pi}}$$

$$\Rightarrow \boxed{E(x) = \mu}$$

Variance.

$$V(X) = E(X^2) - (E(X))^2 \quad \text{——— (1)}$$

Consider,

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx.$$

$$= \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put $z = \frac{x-\mu}{\sigma} \Rightarrow x - \mu = \sigma z$

$\Rightarrow x = \mu + \sigma z \Rightarrow dx = \sigma dz$

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$$\Rightarrow E(X^2) = \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma z)^2 e^{-\frac{1}{2}z^2} \cancel{\sigma}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu^2 + \sigma^2 z^2 + 2\sigma z) e^{-\frac{1}{2}z^2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mu^2 e^{-\frac{1}{2}z^2} dz + \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{1}{2}z^2} dz$$

$$+ \frac{2\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-\frac{1}{2}z^2} dz$$

$$= \frac{2\mu^2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{1}{2}z^2} dz + \frac{2\sigma^2}{\sqrt{2\pi}} \int_{0}^{\infty} z^2 e^{-\frac{1}{2}z^2} dz$$

$$+ \frac{4\sigma}{\sqrt{2\pi}} (0) \quad \text{odd}$$

$$= \frac{2\mu^2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{1}{2}z^2} dz + \frac{2\sigma^2}{\sqrt{2\pi}} \int_{0}^{\infty} z^2 e^{-\frac{1}{2}z^2} dz$$

Put $\frac{1}{2}z^2 = y \Rightarrow z^2 = 2y$

$\Rightarrow z = \sqrt{2y} \Rightarrow dz = \frac{1}{\sqrt{2y}} dy.$

Then

$$E(X^2) = \frac{2\mu^2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-y} \frac{1}{\sqrt{2y}} dy + \frac{2\sigma^2}{\sqrt{2\pi}} \int_{0}^{\infty} 2y\, e^{-y} \frac{dy}{\sqrt{2y}}$$

$$= \frac{\mu^2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-y} y^{-\frac{1}{2}} dy + \frac{2\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-y} y^{\frac{1}{2}} dy$$

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$$= \frac{\mu^2}{\sqrt{\pi}} \int_0^\infty e^{-y} y^{\frac{1}{2}-1} dy + \frac{2\sigma^2}{\sqrt{\pi}} \int_0^\infty e^{-y} y^{\frac{3}{2}-1} dy$$

$$= \frac{\mu^2}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) + \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right)$$

$$= \frac{\mu^2}{\sqrt{\pi}} \sqrt{\pi} + \frac{2\sigma^2}{\sqrt{\pi}} \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{\mu^2}{\cancel{\sqrt{\pi}}} \cancel{\sqrt{\pi}} + \frac{\cancel{2}\sigma^2}{\cancel{\sqrt{\pi}}} \cdot \frac{\cancel{\sqrt{\pi}}}{\cancel{2}}$$

$$= \mu^2 + \sigma^2$$

Now eq ① becomes

$$V(X) = \cancel{\mu^2} + \sigma^2 - \cancel{\mu^2}$$

$$\Rightarrow \boxed{V(X) = \sigma^2}$$

OR $V(X)$ can also be calculated as

$$V(X) = E(X-\mu)^2$$

$$= \sigma^2$$ already calculated in stand. normal. dis.

③ The median and mode of the normal distribution are each equal to μ, the mean of distribution.

Median: A/2014

The median $= M$ is given by

$$\int_{-\infty}^{M} f(x)\,dx = \frac{1}{2}.$$

Therefore,

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$$\Rightarrow \int_{-\infty}^{M} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}$$

Put $z = \frac{x-\mu}{\sigma} \Rightarrow x-\mu = \sigma z$

$\Rightarrow x = \mu + \sigma z \Rightarrow dx = \sigma dz$

When $x \to -\infty \Rightarrow z \to -\infty$

$x \to M \Rightarrow z \to \frac{M-\mu}{\sigma}$

Then

$$\int_{z=-\infty}^{\frac{M-\mu}{\sigma}} \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \cancel{\sigma} dz = \frac{1}{2}$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{M-\mu}{\sigma}} e^{-\frac{1}{2}z^2} dz = \frac{1}{2}$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{-\frac{1}{2}z^2} dz + \int_{0}^{\frac{M-\mu}{\sigma}} e^{-\frac{1}{2}z^2} dz \right] = \frac{1}{2}$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\frac{1}{2}z^2} dz + \frac{1}{\sqrt{2\pi}} \int_{0}^{\frac{M-\mu}{\sigma}} e^{-\frac{1}{2}z^2} dz = \frac{1}{2} \quad \text{——— (1)}$$

As we know that area under the normal curve is unity.

So $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\Rightarrow \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 1$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz = 1$$

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$$\Rightarrow \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\frac{1}{2}z^2} dz = 1$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\frac{1}{2}z^2} dz = \frac{1}{2}$$

Thus,

$$① \Rightarrow \cancel{\frac{1}{2}} + \frac{1}{\sqrt{2\pi}} \int_{0}^{\frac{M-\mu}{\sigma}} e^{-\frac{1}{2}z^2} dz = \cancel{\frac{1}{2}}$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{0}^{\frac{M-\mu}{\sigma}} e^{-\frac{1}{2}z^2} dz = 0$$

which is only possible if

$$\frac{M-\mu}{\sigma} = 0$$

$$\Rightarrow M-\mu = 0 \quad \Rightarrow \boxed{M = \mu}$$

Mode: A/2014

Here $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

Diff. w.r.t. x, we get

$$f'(x) = \frac{1}{\sigma\sqrt{2\pi}} \left\{ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \left(-\left(\frac{x-\mu}{\sigma}\right)\left(\frac{1}{\sigma}\right) \right) \right\}$$

$$= \frac{-1}{\sigma^3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \left(\frac{x-\mu}{\sigma}\right)$$

$$= \frac{-(x-\mu)}{\sigma^3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Put $f'(x) = 0$.

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Then $\frac{-(x-\mu)}{a^3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{a}\right)^2} = 0$

$\Rightarrow x-\mu = 0 \qquad \because \frac{1}{a^3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{a}\right)^2} \neq 0$

$\Rightarrow \boxed{x = \mu}$

Again diff. w.r.t. x, we get

$$f''(x) = \frac{-1}{a^3\sqrt{2\pi}} \frac{d}{dx}\left[(x-\mu) e^{-\frac{1}{2}\left(\frac{x-\mu}{a}\right)^2}\right]$$

$$= \frac{-1}{a^3\sqrt{2\pi}}\left[(x-\mu) e^{-\frac{1}{2}\left(\frac{x-\mu}{a}\right)^2}\left\{-\left(\frac{x-\mu}{a}\right)\left(\frac{1}{a}\right)\right\} + (1-0) e^{-\frac{1}{2}\left(\frac{x-\mu}{a}\right)^2}\right]$$

$$= \frac{-1}{a^3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{a}\right)^2}\left(\frac{-(\mu-x)^2}{a^2}+1\right)$$

$$= -\frac{1}{a^3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{a}\right)^2}\left\{1-\frac{(x-\mu)^2}{a^2}\right\}$$

Subsituting the value of $x=\mu$ in $f''(x)$, we get

$$f''(x)\Big|_{x=\mu} = \frac{-1}{a^3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\mu-\mu}{a}\right)^2}\{1-0\}$$

$$= \frac{-1}{a^3\sqrt{2\pi}}(1)$$

$$= \frac{-1}{a^3\sqrt{2\pi}} < 0$$

Hence $x=\mu$ is mode.

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S/2018 normal dis is symmetrical

Note: Since mean = median = mode. Therefore the normal distribution is symmetrical and unimodal.

(4) The mean deviation of the normal distribution is approximately $\frac{4}{5}$ of its standard deviation. A/2016, A/2014

Proof.

The mean deviation of the normal r.v. X from the mean, μ, is given by

$$M.D. = E(|X-\mu|)$$

$$= \int_{-\infty}^{\infty} |x-\mu| f(x)\,dx$$

$$= \int_{-\infty}^{\infty} |x-\mu| \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} |x-\mu| e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put $\frac{x-\mu}{\sigma} = z \Rightarrow x-\mu = \sigma z$

$\Rightarrow dx = \sigma dz$

Then

$$M.D. = \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} |\sigma z| e^{-\frac{1}{2}z^2} \cancel{\sigma}\, dz$$

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |z| e^{-\frac{1}{2}z^2} dz \quad \text{even}$$

$$= \frac{2\sigma}{\sqrt{2\pi}} \int_{0}^{\infty} z e^{-\frac{1}{2}z^2} dz$$

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Put $\frac{1}{2}z^2 = y \Rightarrow z^2 = 2y \Rightarrow z\,dz = dy$

Then

$$M.D. = \frac{2\sigma}{\sqrt{2\pi}} \int_0^\infty e^{-y}\,dy$$

$$= \sqrt{\frac{2}{\pi}}\,\sigma \lim_{t\to\infty} \int_0^t e^{-y}\,dy$$

$$= \sqrt{\frac{2}{\pi}}\,\sigma \lim_{t\to\infty} \left[-e^{-y}\right]_0^t$$

$$= -\sqrt{\frac{2}{\pi}}\,\sigma \lim_{t\to\infty} \left(\frac{1}{e^t} - \frac{1}{e^0}\right)$$

$$= -\sqrt{\frac{2}{\pi}}\,\sigma\,(0-1)$$

$$= \sqrt{\frac{2}{\pi}}\,\sigma$$

$$= 0.80\,\sigma$$

$\Rightarrow M.D. = \frac{4}{5}\sigma = \frac{4}{5}$ standard deviation.

A/2010.

(5) The normal curve has points of inflection which are equidistant from the mean.

Proof.

The point of inflection is obtained by

$$f''(x) = 0$$

Here $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

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$f''(x)$ Already obtained in mode

$$\Rightarrow f''(x) = -\frac{1}{\sigma^3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \left\{1 - \frac{(x-\mu)^2}{\sigma^2}\right\}$$

For point of inflection

$$f''(x) = 0$$

$$\Rightarrow -\frac{1}{\sigma^3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \left\{1 - \frac{(x-\mu)^2}{\sigma^2}\right\} = 0$$

$$\Rightarrow 1 - \frac{(x-\mu)^2}{\sigma^2} = 0 \quad \because -\frac{1}{\sigma^3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \neq 0$$

$$\Rightarrow \frac{(x-\mu)^2}{\sigma^2} = 1$$

$$\Rightarrow (x-\mu)^2 = \sigma^2$$

$$\Rightarrow x-\mu = \pm\sigma \quad \Rightarrow x = \mu \pm \sigma$$

$$x = \mu + \sigma \quad \text{or} \quad x = \mu - \sigma$$

$$f(x)\Big|_{x=\mu+\sigma} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\cancel{\mu}+\sigma-\cancel{\mu}}{\sigma}\right)^2}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2}$$

$$= \frac{1}{\sigma\sqrt{2\pi e}}$$

and $$f(x)\Big|_{x=\mu-\sigma} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\cancel{\mu}-\sigma-\cancel{\mu}}{\sigma}\right)^2}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2(-1)^2}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2} = \frac{1}{\sigma\sqrt{2\pi e}}$$

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Hence the two points of inflection of normal curve are

$$\left(\mu-\sigma, \frac{1}{\sigma\sqrt{2\pi e}}\right) \text{ and } \left(\mu+\sigma, \frac{1}{\sigma\sqrt{2\pi e}}\right).$$

6) For the normal distribution, the odd order moments about the mean are all zero and the even order moments are given by

$$\mu_{2n} = (2n-1)(2n-3)\cdots 5\cdot 3\cdot 1\, \sigma^{2n}$$

Proof.

Odd Order Moments.

The odd order moments about mean is given by

$$\mu_{2n+1} = E(X-\mu)^{2n+1}$$

$$= \int_{-\infty}^{\infty} (x-\mu)^{2n+1} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} (x-\mu)^{2n+1} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^{2n+1} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put $z = \frac{x-\mu}{\sigma} \Rightarrow x-\mu = \sigma z$

$\Rightarrow dx = \sigma\, dz$

$$\mu_{2n+1} = \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z)^{2n+1} e^{-\frac{1}{2}z^2} \cancel{\sigma}\, dz$$

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$$= \frac{\sigma^{2n+1}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2n+1} e^{-\frac{1}{2}z^2} dz \quad \because \text{odd function}$$

$$= \frac{2\sigma^{2n+1}}{\sqrt{2\pi}} (0) = 0$$

$$\Rightarrow \boxed{\mu_{2n+1} = 0}$$

Even Order Moments.

The even order moments about mean is given by

$$\mu_{2n} = E(X-\mu)^{2n}$$

$$= \int_{-\infty}^{\infty} (x-\mu)^{2n} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} (x-\mu)^{2n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^{2n} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put $z = \frac{x-\mu}{\sigma} \Rightarrow x-\mu = \sigma z \Rightarrow dx = \sigma dz$

Then

$$\mu_{2n} = \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z)^{2n} e^{-\frac{1}{2}z^2} \cancel{\sigma} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma^{2n} \cdot z^{2n} e^{-\frac{1}{2}z^2} dz$$

$$= \frac{\sigma^{2n}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2n} e^{-\frac{1}{2}z^2} dz$$

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$$= \frac{2a^{2n}}{\sqrt{2\pi}} \int_0^\infty z^{2n} e^{-\frac{1}{2}z^2} dz$$

Let $\frac{1}{2}z^2 = y \Rightarrow z^2 = 2y$

$\Rightarrow z\,dz = dy$

$\Rightarrow dy = z\,dz \Rightarrow dz = \frac{1}{\sqrt{2y}} dy$

$$\Rightarrow \mu_{2n} = \frac{2a^{2n}}{\sqrt{2\pi}} \int_0^\infty (2y)^n e^{-y} \frac{dy}{\sqrt{2y}}$$

$$= \frac{2a^{2n}}{\sqrt{2\pi}} \int_0^\infty (2y)^{n-\frac{1}{2}} e^{-y} dy$$

$$= \frac{2a^{2n}}{\sqrt{2\pi}} \cdot 2^{n-\frac{1}{2}} \int_0^\infty e^{-y} y^{n-\frac{1}{2}} dy$$

$$= \frac{a^{2n}}{\sqrt{2\pi}} 2^{n+\frac{1}{2}} \int_0^\infty e^{-y} y^{(n+\frac{1}{2})-1} dy$$

$$= \frac{a^{2n}}{\sqrt{\pi}} 2^n \Gamma(n+\tfrac{1}{2})$$

$$= \frac{a^{2n}}{\sqrt{\pi}} 2^n (n-\tfrac{1}{2})(n-\tfrac{3}{2})(n-\tfrac{5}{2}) \cdots (\tfrac{5}{2})(\tfrac{3}{2})(\tfrac{1}{2})\Gamma(\tfrac{1}{2})$$

$$= \frac{a^{2n} 2^n}{\cancel{\sqrt{\pi}}} \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\left(\frac{2n-5}{2}\right) \cdots (\tfrac{5}{2})(\tfrac{3}{2})(\tfrac{1}{2}) \cancel{\sqrt{\pi}}$$

$$= a^{2n} \cancel{2^n} \frac{(2n-1)(2n-3)(2n-5) \cdots 5 \cdot 3 \cdot 1}{\cancel{2^n}}$$

$$= (2n-1)(2n-3)(2n-5) \cdots 5 \cdot 3 \cdot 1\, a^{2n}.$$

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Derive the mgf of normal dis. and deduce the mgf of standard normal variate.

M.G.F. About Origin. A/2012

$$M_0(t) = E(e^{tx})$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

Put $z = \frac{x-\mu}{\sigma} \Rightarrow x-\mu = \sigma z$

$\Rightarrow x = \mu + \sigma z \Rightarrow dx = \sigma dz.$

$$\Rightarrow M_0(t) = \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} e^{-\frac{1}{2}z^2} \cancel{\sigma} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} e^{-\frac{1}{2}z^2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mu t} \cdot e^{\sigma z t} e^{-\frac{1}{2}z^2} dz.$$

$$= \frac{1}{\sqrt{2\pi}} e^{\mu t} \int_{-\infty}^{\infty} e^{\sigma z t - \frac{1}{2}z^2} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2\sigma z t)} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[z^2 - 2\sigma z t + (\sigma t)^2 - (\sigma t)^2\right]} dz$$

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$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[(z-at)^2 - a^2t^2]} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-at)^2} e^{\frac{1}{2}a^2t^2} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} e^{\frac{1}{2}a^2t^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-at)^2} dz$$

Put $z - at = u$

$\Rightarrow dz = du$

$$\Rightarrow M_0(t) = \frac{e^{\mu t + \frac{1}{2}a^2t^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} du$$

$$= \frac{2e^{\mu t + \frac{1}{2}a^2t^2}}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{1}{2}u^2} du$$ {even function

Put

$\frac{1}{2}u^2 = y \Rightarrow u^2 = 2y \Rightarrow u = \sqrt{2y}$

$\Rightarrow du = \frac{1}{\sqrt{2y}} dy$

Then

$$M_0(t) = \frac{2e^{\mu t + \frac{1}{2}a^2t^2}}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-y} \frac{dy}{\sqrt{2y}}$$

$$= \frac{e^{\mu t + \frac{1}{2}a^2t^2}}{\sqrt{\pi}} \int_{0}^{\infty} e^{-y} y^{-1/2} dy$$

$$= \frac{e^{\mu t + \frac{1}{2}a^2t^2}}{\sqrt{\pi}} \int_{0}^{\infty} e^{-y} y^{1/2 - 1} dy$$

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$$= \frac{e^{\mu t + \frac{1}{2}a^2t^2}}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) = \frac{e^{\mu t + \frac{1}{2}a^2t^2}}{\cancel{\sqrt{\pi}}} \cancel{\sqrt{\pi}}$$

$$\Rightarrow \boxed{M_0(t) = e^{\mu t + \frac{1}{2}a^2t^2}}$$

for standard normal variate

$M_0(t) = e^{\frac{1}{2}t^2}$

as $\mu = 0$, $a^2 = 1$

$$\mu_1' = \frac{d^1}{dt^1} e^{\mu t + \frac{1}{2}a^2t^2}\Big|_{t=0}$$

$$\mu_1' = \frac{d}{dt} e^{\mu t + \frac{1}{2}a^2t^2}\Big|_{t=0}$$

$$= e^{\mu t + \frac{1}{2}a^2t^2}(\mu + a^2 t)\Big|_{t=0}$$

$$\Rightarrow \boxed{\mu_1' = \mu}$$

$$\mu_2' = \frac{d}{dt}\left[e^{\mu t + \frac{1}{2}a^2t^2}(\mu + a^2 t)\right]_{t=0}$$

$$= \left[e^{\mu t + \frac{1}{2}a^2t^2}(\mu + a^2t)(\mu + a^2t) + e^{\mu t + \frac{1}{2}a^2t^2}(a^2)\right]_{t=0}$$

$$= \left[e^{\mu t + \frac{1}{2}a^2t^2}\left\{(\mu + a^2t)^2 + a^2\right\}\right]_{t=0}$$

$$\Rightarrow \boxed{\mu_2' = \mu^2 + a^2}$$

$$\mu_3' = \frac{d}{dt}\left[e^{\mu t + \frac{1}{2}a^2t^2}\left\{(\mu + a^2t)^2 + a^2\right\}\right]_{t=0}$$

$$= \left[e^{\mu t + \frac{1}{2}a^2t^2}(\mu + a^2t)\left\{(\mu + a^2t)^2 + a^2\right\} + e^{\mu t + \frac{1}{2}a^2t^2}\left\{2(\mu + a^2t)(a^2)\right\}\right]_{t=0}$$

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$$= \left[e^{\mu t + \frac{1}{2}a^2t^2} \left\{ (\mu + a^2 t)^3 + 3a^2 (\mu + a^2 t) \right\} \right]_{t=0}$$

$$\Rightarrow \boxed{\begin{aligned} \mu_3' &= \mu^3 + 3a^2\mu \\ &= \mu(\mu^2 + 3a^2) \end{aligned}}$$

$$\mu_4' = \left[e^{\mu t + \frac{1}{2}a^2t^2} \left\{ 3(\mu + a^2 t)^2 (a^2) + 3a^2 (a^2) \right\} + e^{\mu t + \frac{1}{2}a^2t^2} (\mu + a^2 t) \left\{ (\mu + a^2 t)^3 + 3a^2 (\mu + a^2 t) \right\} \right]_{t=0}$$

$$= 3(\mu)^2 a^2 + 3a^4 + (\mu)(\mu^3 + 3a^2\mu)$$

$$= 3\mu^2 a^2 + 3a^4 + \mu^4 + 3a^2\mu^2$$

$$\Rightarrow \boxed{\mu_4' = \mu^4 + 3a^4 + 6\mu^2 a^2}$$

M.G.F. About Mean.

$$M_\mu(t) = E(e^{t(x-\mu)})$$

where μ is the mean.

$$\Rightarrow M_\mu(t) = \int_{-\infty}^{\infty} e^{t(x-\mu)} f(x)\,dx$$

$$= e^{-\mu t} \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= e^{-\mu t} M_0(t)$$

$$= e^{-\mu t} \cdot e^{\mu t + \frac{1}{2}a^2t^2}$$

$$\Rightarrow \boxed{M_\mu(t) = e^{\frac{1}{2}a^2t^2}}$$

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$$\mu_s = \frac{d^s}{dt^s} e^{\frac{1}{2}a^2t^2}\Big|_{t=0}$$

$$s=1 \Rightarrow \mu_1 = \frac{d}{dt} e^{\frac{1}{2}a^2t^2}\Big|_{t=0}$$

$$= e^{\frac{1}{2}a^2t^2} a^2t\Big|_{t=0}$$

$$\Rightarrow \boxed{\mu_1 = 0}$$

$$\mu_2 = a^2 e^{\frac{1}{2}a^2t^2} t\Big|_{t=0}$$

$$= a^2\left\{t e^{\frac{1}{2}a^2t^2} a^2t + e^{\frac{1}{2}a^2t^2}\right\}_{t=0}$$

$$= a^2\left\{t^2a^2 e^{\frac{1}{2}a^2t^2} + e^{\frac{1}{2}a^2t^2}\right\}_{t=0}$$

$$= a^2(0+1)$$

$$\Rightarrow \boxed{\mu_2 = a^2}$$

$$\mu_3 = a^2\left\{2t e^{\frac{1}{2}a^2t^2} + t^2a^2 e^{\frac{1}{2}a^2t^2} a^2t + e^{\frac{1}{2}a^2t^2} a^2t\right\}.$$

$$= a^2\left\{2ta^2 + t^3a^4 + a^2t\right\} e^{\frac{1}{2}a^2t^2}\Big|_{t=0}$$

$$\Rightarrow \boxed{\mu_3 = 0}$$

$$\mu_4 = a^2\frac{d}{dt}\left\{2ta^2 + t^3a^4 + a^2t\right\} e^{\frac{1}{2}a^2t^2}\Big|_{t=0}$$

$$= a^2\left[2a^2 + 3t^2a^4 + a^2\right] e^{\frac{1}{2}a^2t^2}$$

$$+ e^{\frac{1}{2}a^2t^2}\left[a^2t\right)\left\{2ta^2 + t^3a^4 + a^2t\right\}\Big]_{t=0}$$

$$= a^2(2a^2 + a^2) = 3a^4 \Rightarrow \boxed{\mu_4 = 3a^4}$$

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Cumulant Generating Function.

The cumulant g.f. is

$$K(t) = \ln M_0(t)$$
$$= \ln e^{\mu t + \frac{1}{2}a^2t^2}$$
$$= \mu t + \frac{1}{2}a^2t^2$$

$$\Rightarrow K(t) = \mu \frac{t}{1!} + a^2 \frac{t^2}{2!} \quad \text{——— (1)}$$

The general expression for cumulant generating function is

$$K(t) = K_1 \frac{t}{1!} + K_2 \frac{t^2}{2!} + K_3 \frac{t^3}{3!} + K_4 \frac{t^4}{4!} + \cdots \quad \text{——— (2)}$$

Comparing (1) and (2), we get

$K_1 = \mu$; $K_2 = a^2$

$K_3 = 0$; $K_4 = 0$

$\Rightarrow K_s = 0$ for $s \geq 3$.

Question 1. A/2014, A/2011,

Normal Approximation to the binomial Distribution. A/2018: Prove that normal dis. is A/2017 a limiting case of B. dis.

Sol.

The probability for a binomial r.v. X to take the value x is

$$f(x) = \binom{n}{x} p^x q^{n-x} \; ; \quad x = 0, 1, 2, \ldots n$$

The variable X has mean $\mu = np$ and

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$V(X) = \sigma^2 = npq.$

To derive the limiting form, we define a new random variable Z by the relation

$$Z = \frac{X - \mu}{\sigma}$$

$$\Rightarrow Z = \frac{X - np}{\sqrt{nqp}}$$

Obviously $E(Z) = 0$ and $V(Z) = 1$ (already calculated)

We shall use the moment generating function which is given by

$$M(t) = E(e^{tZ})$$

$$= E\left(e^{t\frac{(x-np)}{\sqrt{npq}}}\right)$$

$$= \sum_{x=0}^{n} e^{t(x-np)/\sqrt{npq}} \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} e^{(tx - npt)/\sqrt{npq}} \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} e^{tx/\sqrt{nqp}} e^{-npt/\sqrt{npq}} \binom{n}{x} p^x q^{n-x}$$

$$= e^{-npt/\sqrt{npq}} \sum_{x=0}^{n} e^{tx/\sqrt{npq}} \binom{n}{x} p^x q^{n-x}$$

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$$= e^{-npt/\sqrt{npq}} \sum_{x=0}^{n} \frac{n!}{x!\,(n-x)!} \left(pe^{t/\sqrt{npq}}\right)^{x} q^{n-x}$$

$$= e^{-npt/\sqrt{npq}}\, n! \left\{ \frac{1}{0!\,n!} \left(pe^{t/\sqrt{npq}}\right)^{0} q^{n} + \frac{1}{1!\,(n-1)!} \left(pe^{t/\sqrt{npq}}\right)^{1} q^{n-1} + \frac{1}{2!\,(n-2)!} \left(pe^{t/\sqrt{npq}}\right)^{2} q^{n-2} + \cdots + \frac{1}{n!\,(0)!} \left(pe^{t/\sqrt{npq}}\right)^{n} q^{0} \right\}$$

$$= e^{-npt/\sqrt{npq}} \left\{ q^{n} + n\left(pe^{t/\sqrt{npq}}\right) q^{n-1} + \frac{n(n-1)}{2!} \left(pe^{t/\sqrt{npq}}\right)^{2} q^{n-2} + \cdots \frac{n!}{n!} \left(pe^{t/\sqrt{npq}}\right)^{n} q^{0} \right\}$$

$$= e^{-npt/\sqrt{npq}} \left(q + pe^{t/\sqrt{npq}}\right)^{n}$$ by binomial theorem.

$$= \left[e^{-pt/\sqrt{npq}} \left(q + pe^{t/\sqrt{npq}}\right) \right]^{n}$$

$$= \left[q\, e^{-pt/\sqrt{npq}} + p\, e^{(1-p)t/\sqrt{npq}} \right]^{n}$$

$$= \left[q e^{-pt/\sqrt{npq}} + p e^{qt/\sqrt{npq}} \right]^{n} \quad \text{——— (1)}$$

Since $e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

and $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$

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Therefore,

$$e^{qt/\sqrt{npq}} = 1 + \frac{q}{\sqrt{npq}}t + \frac{(qt/\sqrt{npq})^2}{2!} + \frac{(qt/\sqrt{npq})^3}{3!} + \cdots$$

$$\Rightarrow pe^{qt/\sqrt{npq}} = p + \frac{pq}{\sqrt{npq}}t + p\left(\frac{q}{\sqrt{npq}}\right)^2\frac{t^2}{2!} + p\left(\frac{q}{\sqrt{npq}}\right)^3\frac{t^3}{3!} + \cdots \quad \text{——— (2)}$$

and

$$e^{-pt/\sqrt{npq}} = 1 - \frac{p}{\sqrt{npq}}t + \frac{(-pt/\sqrt{npq})^2}{2!} + \frac{(-pt/\sqrt{npq})^3}{3!} + \cdots$$

$$\Rightarrow qe^{-pt/\sqrt{npq}} = q - \frac{pq}{\sqrt{npq}}t + q\left(\frac{p}{\sqrt{npq}}\right)\frac{t^2}{2!} + q\left(\frac{p}{\sqrt{npq}}\right)^3\frac{t^3}{3!} + \cdots \quad \text{——— (3)}$$

Adding (2) and (3), we get

$$qe^{-pt/\sqrt{npq}} + pe^{qt/\sqrt{npq}} = p+q + \left\{p\left(\frac{q}{\sqrt{npq}}\right)^2 + q\left(\frac{p}{\sqrt{npq}}\right)^2\right\}\frac{t^2}{2!}$$

$$+\left\{p\left(\frac{q}{\sqrt{npq}}\right)^3 - q\left(\frac{p}{\sqrt{npq}}\right)^3\right\}\frac{t^3}{3!} + \cdots$$

$$= 1 + \left(\frac{pq^2}{npq} + \frac{qp^2}{npq}\right)\frac{t^2}{2!} + \left(\frac{pq^3}{npq\sqrt{npq}} - \frac{qp^3}{npq\sqrt{npq}}\right)\frac{t^3}{3!} + \cdots$$

$$= 1 + \left(\frac{q}{n} + \frac{p}{n}\right)\frac{t^2}{2!} + \left(\frac{q^2}{n\sqrt{npq}} - \frac{p^2}{n\sqrt{npq}}\right)\frac{t^3}{3!} + \cdots$$

$$= 1 + \left(\frac{q+p}{n}\right)\frac{t^2}{2!} + \left(\frac{q^2-p^2}{n\sqrt{npq}}\right)\frac{t^3}{3!} + \cdots$$

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$$= 1 + \frac{t^2}{n\,2!} + \frac{q-p}{n\sqrt{npq}}\frac{t^3}{3!} + \cdots$$

Now equation ① becomes

$$M(t) = \left(1 + \frac{1}{n}\cdot\frac{t^2}{2!} + \frac{q-p}{n\sqrt{npq}}\frac{t^3}{3!} + \cdots\right)^n$$

Taking ln on both sides, we get

$$\ln M(t) = \ln\left(1 + \frac{1}{n}\frac{t^2}{2!} + \frac{q-p}{n\sqrt{npq}}\frac{t^3}{3!} + \cdots\right)^n$$

$$= n\ln\left(1 + \frac{1}{n}\frac{t^2}{2!} + \frac{q-p}{n\sqrt{npq}}\frac{t^3}{3!} + \cdots\right)$$

Since $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

Therefore,

$$\ln M(t) = n\left[\left(\frac{1}{n}\cdot\frac{t^2}{2!} + \frac{q-p}{n\sqrt{npq}}\frac{t^3}{3!} + \cdots\right) - \frac{1}{2}\left(\frac{1}{n}\cdot\frac{t^2}{2!} + \frac{q-p}{n\sqrt{npq}}\frac{t^3}{3!} + \cdots\right)^2\right.$$

$$\left. + \frac{1}{3}\left(\frac{1}{n}\cdot\frac{t^2}{2!} + \frac{q-p}{n\sqrt{npq}}\frac{t^3}{3!} + \cdots\right)^3 - \frac{1}{4}\left(\frac{1}{n}\cdot\frac{t^2}{2!} + \frac{q-p}{n\sqrt{npq}}\frac{t^3}{3!} + \cdots\right)^4 + \cdots\right]$$

$$= n\left[\frac{1}{n}\frac{t^2}{2!} + \frac{q-p}{n\sqrt{npq}}\frac{t^3}{3!} - \frac{1}{2}\left(\frac{1}{n^2}\frac{t^4}{4} + \left(\frac{q-p}{n\sqrt{npq}}\frac{t^3}{3!}\right)^2 + \cdots\right) + \cdots\right]$$

$$= n\left(\frac{1}{n}\frac{t^2}{2!} + \frac{q-p}{n\sqrt{npq}}\frac{t^3}{3!} + \cdots\right)$$

$$= \frac{t^2}{2!} + \frac{q-p}{\sqrt{npq}}\frac{t^3}{3!} + \cdots$$

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Taking $\lim_{n\to\infty}$, we have

$$\lim_{n\to\infty} \log_e M(t) = \lim_{n\to\infty} \left(\frac{t^2}{2} + \frac{q-p}{\sqrt{npq}} \frac{t^3}{3!} + \cdots \right)$$

$$= t^2/2 + 0$$

$$\Rightarrow \lim_{n\to\infty} \ln M(t) = t^2/2$$

Taking anti-ln, we get

$$\lim_{n\to\infty} M(t) = e^{t^2/2} \quad \text{——— (4)}$$

The mgf for normal distribution is

$$M(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

But for standarized normal distribution

$\mu = 0$, $\sigma^2 = 1$

$$M(t) = e^{\frac{1}{2}t^2} \quad \text{——— (5)}$$

From (4) and (5), we have

$$\lim_{n\to\infty} b(x; n, p) = N(x; np, npq)$$

Hence proved.

Question 2.

Normal Approximation to the Poisson distribution.

Sol.

The probability for a Poisson r.v. X to take the value x is

$$f(x) = e^{-\mu} \frac{\mu^x}{x!} \; ; x = 0, 1, 2, \ldots \infty$$

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The variable X has has mean μ and variance μ.

To derive the limiting form, we define a new r.v. Z by the relation

$$Z = \frac{X-\mu}{\sigma}$$

$$\Rightarrow Z = \frac{X-\mu}{\sqrt{\mu}} \qquad \because \sigma^2 = \mu$$

Obviously $E(Z)=0$ and $V(Z)=1$.

We shall use m.g.f. which is given by

$$M(t) = E(e^{tZ})$$

$$= E\left(e^{t\left(\frac{x-\mu}{\sqrt{\mu}}\right)}\right)$$

$$= \sum_{x=0}^{\infty} e^{t\left(\frac{x-\mu}{\sqrt{\mu}}\right)} \frac{e^{-\mu}\mu^x}{x!}$$

$$= \sum_{x=0}^{\infty} e^{(tx-\mu t)/\sqrt{\mu}} \frac{e^{-\mu}\mu^x}{x!}$$

$$= \sum_{x=0}^{\infty} e^{tx/\sqrt{\mu}} \cdot e^{-\mu t/\sqrt{\mu}} \frac{e^{-\mu}\mu^x}{x!}$$

$$= e^{-\mu t/\sqrt{\mu} - \mu} \sum_{x=0}^{\infty} e^{tx/\sqrt{\mu}} \frac{\mu^x}{x!}$$

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$$= e^{-\mu\left(\frac{t}{\sqrt{\mu}}+1\right)} \sum_{x=0}^{\infty} \frac{\left(\mu e^{t/\sqrt{\mu}}\right)^x}{x!}$$

$$= e^{-\mu\left(\frac{t}{\sqrt{\mu}}+1\right)} \left\{ 1 + \frac{\left(\mu e^{t/\sqrt{\mu}}\right)}{1!} + \frac{\left(\mu e^{t/\sqrt{\mu}}\right)^2}{2!} + \frac{\left(\mu e^{t/\sqrt{\mu}}\right)^3}{3!} + \cdots \right\}$$

$$= e^{-\mu\left(\frac{t}{\sqrt{\mu}}+1\right)} e^{\mu e^{t/\sqrt{\mu}}}$$

$$= e^{-\mu\left(\frac{t}{\sqrt{\mu}}+1-e^{t/\sqrt{\mu}}\right)}$$

$$= e^{-t\sqrt{\mu}-\mu+\mu e^{t/\sqrt{\mu}}}$$

Taking ln on both sides, we get

$$\ln M(t) = \ln e^{-t\sqrt{\mu}-\mu+\mu e^{t/\sqrt{\mu}}}$$

$$= -t\sqrt{\mu} - \mu + \mu e^{t/\sqrt{\mu}}$$

$$= -t\sqrt{\mu} - \mu + \mu\left\{1 + \frac{t}{\sqrt{\mu}} + \frac{(t/\sqrt{\mu})^2}{2!} + \frac{(t/\sqrt{\mu})^3}{3!} + \cdots\right\}$$

$$= -t\sqrt{\mu} \cancel{-\mu} \cancel{+\mu} + \frac{\mu t}{\sqrt{\mu}} + \frac{\mu}{2!}\left(\frac{t}{\sqrt{\mu}}\right)^2 + \frac{\mu}{3!}\left(t/\sqrt{\mu}\right)^3 + \cdots$$

$$= \cancel{-t\sqrt{\mu}} \cancel{+t\sqrt{\mu}} + \frac{\cancel{\mu}}{2}\frac{t^2}{\cancel{\mu}} + \frac{\cancel{\mu}}{3!}\frac{t^3}{\cancel{\mu}\sqrt{\mu}} + \cdots$$

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$$\Rightarrow \ln M(t) = \frac{t^2}{2} + \frac{1}{\sqrt{\mu}}\frac{t^3}{3!} + \cdots$$

Taking $\lim_{\mu \to \infty}$, we get

$$\Rightarrow \lim_{\mu \to \infty} \ln M(t) = \lim_{\mu \to \infty} \left(\frac{t^2}{2} + \frac{1}{\sqrt{\mu}}\frac{t^3}{3!} + \cdots \right)$$

$$\Rightarrow \lim_{\mu \to \infty} \ln M(t) = \frac{t^2}{2}$$

$$\Rightarrow \lim_{\mu \to \infty} M(t) = e^{t^2/2} \quad \text{——— (1)}$$

The mgf for normal distribution is

$$M(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

But for standardized normal distri.

$\mu = 0$ and $\sigma^2 = 1$.

So $$M(t) = e^{\frac{1}{2}t^2} \quad \text{——— (2)}$$

From (1) and (2), we have

$$\lim_{\mu \to \infty} p(x; \mu) = N(x; \mu, \mu).$$

Hence Poisson distribution approaches the normal distribution as $\mu \to \infty$.

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Question 3.

If X and Y have joint p.d.f.

$$f(x,y) = e^{-(x+y)} \frac{x^3 y^4}{\Gamma(4)\Gamma(5)}, \quad x>0, y>0$$

and '0' elsewhere, then find

(i) P.d.f. of $U = \frac{X}{X+Y}$

(ii) $E(U)$ (iii) $Var(U)$

Sol. (i)

Given that

$$f(x,y) = \frac{1}{\Gamma(4)\Gamma(5)} e^{-(x+y)} x^3 y^4$$

Probability distribution of $U = \frac{X}{X+Y}$

i.e. $u = \frac{x}{x+y}$ is given by

$$P\left(\frac{X}{X+Y} \le u\right) = \iint_{\frac{x}{x+y} \le u} \frac{1}{\Gamma(4)\Gamma(5)} e^{-(x+y)} x^{4-1} y^{5-1} \, dx\,dy$$

Proceed as Q_6 (ii) by replacing $\mu = 4$ and $\nu = 5$ and finally we get

$$P\left(\frac{X}{X+Y} \le u\right) = \frac{1}{\beta(4,5)} \int_0^1 t^{4-1} (1-t)^{5-1} \, dt$$

$$\Rightarrow g(t) = \frac{t^3 (1-t)^4}{\beta(4,5)}$$ is req. p.d.f.

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(ii)

$$E(U) = \int_{-\infty}^{\infty} t\, g(t)\, dt$$

$$= \int_0^1 t \frac{1}{\beta(4,5)} t^3 (1-t)^4 dt$$

$$= \frac{1}{\beta(4,5)} \int_0^1 t^4 (1-t)^4 dt.$$

$$= \frac{1}{\beta(4,5)} \int_0^1 t^{5-1} (1-t)^{5-1} dt$$

$$= \frac{1}{\beta(4,5)} \cdot \beta(5,5)$$

Since

$$\beta(\mu,\nu) = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}$$

Therefore,

$$E(U) = \frac{\Gamma(4+5)}{\Gamma(4)\cancel{\Gamma(5)}} \times \frac{\Gamma(5)\cancel{\Gamma(5)}}{\Gamma(5+5)}$$

$$= \frac{\Gamma(9)}{\Gamma(4)} \cdot \frac{\Gamma(5)}{\Gamma(10)}$$

$$= \frac{\cancel{\Gamma(9)}}{\cancel{\Gamma(4)}} \frac{4\cancel{\Gamma(4)}}{9\cancel{\Gamma(9)}}$$

$$= 4/9$$

$$\Rightarrow E(U) = 4/9$$

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(iii)

$$Var(U) = E(U^2) - (E(U))^2 \quad \text{——— } (S)$$

Consider

$$E(U^2) = \int_{-\infty}^{\infty} u^2 g(u)\, du$$

{dummy variable}

$$= \int_0^1 u^2 \frac{1}{\beta(4,5)} u^3 (1-u)^4 du$$

$$= \frac{1}{\beta(4,5)} \int_0^1 u^5 (1-u)^4 du$$

$$= \frac{1}{\beta(4,5)} \int_0^1 u^{6-1} (1-u)^{5-1} du$$

$$= \frac{1}{\beta(4,5)} \cdot \beta(6,5)$$

$$= \frac{\Gamma(4+5)}{\Gamma(4)\cancel{\Gamma(5)}} \cdot \frac{\Gamma(6)\cancel{\Gamma(5)}}{\Gamma(6+5)}$$

$$= \frac{\Gamma(9)}{\Gamma(4)} \cdot \frac{\Gamma(6)}{\Gamma(11)}$$

$$= \frac{\cancel{\Gamma(9)}}{\cancel{\Gamma(4)}} \cdot \frac{(5)(4)\cancel{\Gamma(4)}}{(10)(9)\cancel{\Gamma(9)}}$$

$$= \frac{20}{90} = \frac{2}{9}$$

Thus

$$(S) \Rightarrow Var(U) = \frac{2}{9} - \frac{16}{81} = \frac{18-16}{81}$$

$$= \frac{2}{81}$$

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